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Stability of Collinear Libration Points in the Photogravitaional Elliptic Restricted Three Body Problem (ER3BP) under the Effects of Pr-Drag Force and Triaxialityenclosed by a Belt

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

The impact of radiation pressure, Poynting-Robertson Drag (Pr-drag) force and triaxiality of the main bodies surrounded by a belt (circumbinary disc) on the positions and stability of a third body of an infinestisimal mass aroud the collinear Lagrangian points in the elliptic restricted three body problem (ER3BP) are studied. We have obtained the solutions to the location of collinear Lagrangian points L_i (i =1,2,3). In order to investigate the stability of the model two binary system (FL virginis and Procyon) were employed. It was found that the positions and stability of the libration points are affected by trixiality, Pr-drag force and the gravitational pontetial from the belt. The collinear libration points were found to be unstable.

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1. INTRODUCTION

The solution totwo-body problem have been found, but the solution to three- body problem is yet unobtainable when three bodies are heavenly bodies for example earth, moon and sun. For a close close form solution to be found Euler in 1765 introduced the restricted three body problem, a simplification of the three body problem where the small body is assumed to have a negligible mass compared to the two other larger bodies and rotatesin the plane defined by the two main bodies aound their common centre of mass. An example of the restricted three body problem is earth, moon and an artificial satellite.

During the classical era of investigation of the restricted three body problem Euler in 1765 and Lagrange in 1772 obtained a number of particular solutions from the revolving frame where the infinitesimal mass has zero velocity and acceleration. The solutions obtained are found to be the equilibrium positions, which are five in the R3BP and are called Lagrangian equilibrium points: the first three (L_1,L_2, L_3) are called collinear equilibrium (libration) points and the remaining two are triangular equilibrium (libration) points named Langragian equilibrium points (L_4 and L_5).The collinear equilibrium pointsare located on the line joining the primaries.

For example in the earth –sun system in which the earth orbits the sun, the three collinear points lie on the line joining the earth and the sun.$L_1$is between sun and the earth,L_2 in the same direction as the earth,$L_3$is opposite the earth. The collinear points are unstable and any object such as satellite placed at this points cannot remain in position except some thrusters are attached. This has allowed SOHO's satellite to be kept at $L_1$to monitor the sun and NASA satellite called WMAP to be stationed at L_2. If trusthers are not attached, several forces such as gravitational, corolis force and centrifugal forces can drift the satellite from position.

Apart from these forces other forces such as radiation pressure force, main body shapes, circumbinary disc and P-R drag force can affect the position of the third body at equilibrium points. According to Singh [1] there are several forces in the solar system as such it is inappropriate to consider gravitational force alone when studying the dynamics of a solar system. Radzievskii [2] pointed out that "the repelling forces of radiation pressure is dorminant when a star collides with a gas and dust particles apart from the gravitational force". The forces have substantial effect on the stability of equilibrium points. In particular, the radiation pressure force which is the major force acting on planetary objects after gravity. The classical R3BP is inadequate in describing the dynamics of a particle emitting radiation therefore in other to account for this force in the equations of motion the classical potential function was amended to admit it. Thus the problem becomes generalized. Kumar and Ishwar [3] included radiation pressure in their study. Singh [1] study "the

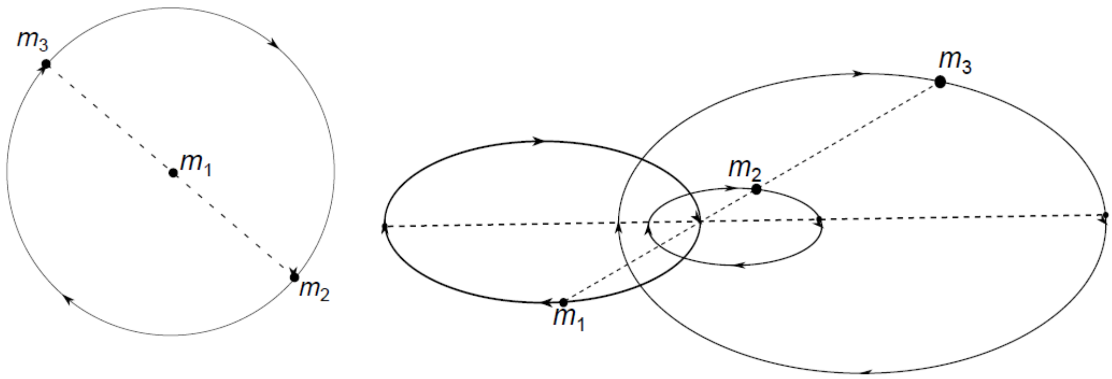


Fig. 1. The euler solution: The three bodies remain collinear at all times, in elliptical orbits around the centre of mass. Left: All masses equal. Right: Unequal masses

effect small perturbations of the Corolis and centrifugal forces, triaxiality and radiation pressure of the primaries on the triangular libration points".

The generalization attracted a lot of researchers to study the effect of the pertubations on the location and stability of R3BP under different characterizations. The pertubations changes the position of the third-body (infinitesimal mass) at equilibrium points slightly and if it departs from the equilibrium points significantly it may not return which may result to instability and the equilibrium points is said to be unstable. If however, the body returns to its original position after departure, such a position of equilibrium point is said to be stable.

"The effect of triaxiality of the bigger primary and oblateness of the companion on the location and stability of the collinear equilibrium pointsin ER3BP have been studied by Singh and Umar" [4]. They observed that the perturbations changes the locations and stability of the collinear libration points due to the effect of eccentricity and the semi-major axis of the primaries and are unstable. Recently, Vicent et al. [5] shows that pertubations can lead to increase in number of collinear points. Four additional collinear equilibrium points L_{ni} = 1, 2, 3, 4, in addition to the three Eulerian points L_{ni} = 1, 2, 3, of the classical case were obtained-a total of seven collinear points. It was observed that out of the four collinear equilibrium points two:L_{n1} and L_{n2} are due to the effect of potential from the belt, while L_{n3} and L_{n4} arise from the effect of triaxiality-in a generalized restricted three-body problem (R3BP) with an oblate-infinitesimal body and triaxial- radiating primaries. [6] explored "vinti's method tolocate the collinear points expressed in the form of infinite series in the elliptical restricted three body problem when both the primaries are radiating and oblate it is found that all the three collinear equilibrium points are unstable". Using two binary systems Luyten-726 and Kruger-60 a defining parameter was established for possible range of stability. "Using analytical method the effect of the radiation pressure, forces due to stellar wind and Poynting-Robertson drag on the dust particles are studied" by [7].The five classical equilibrium points (L_1, L_2, L_3, L_4, L_5) are obtained.It is seen that the collinear equilibrium points (L_1, L_2, L_3) are shifted along the y-coordinate and do not lie on the line joining the primaries due to the effects. In their paper [8] examines the equilibrium points and stability of an oblate-spheroidal infinitesimal mass under the influence of a gravitational radiating primary and an oblate-shaped secondary using the model of the restricted three-body problem. Five collinear equilibrium points are obtained due to the presence of oblateness of either the second primary or the test particle. The collinear points are tested for stability by inspecting the root of the characteristic equation of the system it was found that collinear points are unstable due to the presence of a positive root.The existence of collinear equilibrium points, their solution and stability analysis in the Chermnykh-Like problem under the effect of radiation pressure, oblateness and a discwas investigated by [9] and it shows that "the presence of the disc, creates a new collinear equilibrium point in addition to the three points of the classical problem. For the linear stability analysis of the collinear equilibrium points the disc's outer radius b instead of mass parameter µ was used and it is found that all the collinear equilibrium points are unstable except L3 which is stable for b ∊ (1.3312, 1.5275) provided that remaining parameters are fixed".

"The Poynting-Robertson (P-R) effect also called Poynting-Robertson (P-R) Drag force was named after .John Henry Poynting and Howard Percy Robertson". [10] described the effect based on a theory that supersede the theory of relativity. Later, [11] described the effect in terms of general relativity. The P-R Drag force is a component of radiation pressure and is tangential to the grain's motion.It is an effective force that opposes the direction of the dust grain's motion and causes a drop in the grain's angular momentum. An investigation by [12] reveals that the P-R drag forceis the main force causing instability at the equilibrium point and not the radiation pressure, oblateness and centrifugal forcealthough they also decrease the region of stability.He showed that in the region when motion around the triangular points are stable any slight increase of the P-R drag of the bigger primary by an almost negligible value could result to an unstable equilibrium point overrides other effects and changes stability to instability.

In this work we wish to study the positions and stability of collinear equilibrium points, when the primaries are triaxial,emmiting radiation and surrounded by circumbinary disc (belt) with the smaller primary having an effective P-R Drag force.Two radiating binary stars (FLvirginis and Procyon*)* see [13] Singh and Isah (2021). The paper is an extension of the works of [13] Singh and Isah (2021). We found that adding PR-Drag force to the previous model [13] Singh and Isah (2021) has little or no effect on the system dynamics. The paper is organized as follows: section 1 is introduction, in section 2 the equations of motion are described in detail, while positions of equilibrium points and analysis of their stability can be found in section 3, section 4contains the numerical study of the system finally, discussion and conclusion are presented in section 5.

2. EQUATION OF MOTION

The equations of motion of the infinitesimal mass in the three-dimensional restricted three-body problem with the origin at the center of mass, in a barycentric rotating (also called synodical) coordinate system under the gravitational influence of two radiating triaxial bodies with the P–R drag present and surrounded by belt have the form:

$$\Omega_\xi = (1-e^2)^{-\frac{1}{2}}\Big[\xi - \frac{1}{n^2}\Big\{\frac{(1-\mu)(\xi+\mu)}{r_1{}^3}q_1 + \frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}q_1 - \frac{15(1-\mu)(\xi+\mu)(\sigma_1-\sigma_2)}{2r_1{}^7}q_1\eta^2 -$$

151−μξ+μσ12r17q1ζ2+μξ+μ−1r23q2+3μξ+μ−12σ3−σ42r25q2−15μξ+μ−1σ3−σ42r27q2η2−15μξ+μ−1σ32r27q2ζ2+Mbξr2+T232−W2n2r22ξ+μ−1ξ+μ−1ξ'+ηη'+ζζ'r22+ξ'−nη

(2)

$$\Omega_\eta = (1-e^2)^{-\frac{1}{2}}\Big[\eta\Big\{1 - \frac{1}{n^2}\Big(\frac{(1-\mu)}{r_1{}^3}q_1 + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}q_1 + \frac{3(1-\mu)(\sigma_1-\sigma_2)}{r_1{}^5}q_1 - \frac{15(1-\mu)(\xi+\mu)(\sigma_1-\sigma_2)}{2r_1{}^7}q_1\eta^2 -$$

151−μσ12r17q1ζ2+μr23q2+3μ2σ3−σ42r25q2+3μσ3−σ4r25q2−15μσ3−σ42r27q2η2−15μσ32r27q2ζ2+Mbr2+T232−W2n2r22ηξ+μ−1ξ'+ηη'+ζζ'r22+η'+nξ+μ−1

(3)

$$\Omega_\zeta = (1-e^2)^{-\frac{1}{2}}\Big[-\frac{\zeta}{n^2}\Big(\frac{(1-\mu)}{r_1{}^3}q_1 + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1{}^5}q_1 + \frac{3(1-\mu)\sigma_1}{r_1{}^5}q_1 - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1{}^7}q_1\eta^2 -$$

151−μσ12r17q1ζ2+μr23q2+3μ2σ3−σ42r25q2+3μσ3r25q2−15μσ3−σ42r27q2η2−15μσ32r27q2ζ2−W2n2r22ζξ+μ−1ξ'+ηη'+ζζ'r22+ζ'

(4)

$$r_1^2 = (\xi+\mu)^2 + \eta^2 + \zeta^2, r_2^2 = (\xi+\mu-1)^2 + \eta^2 + \zeta^2 \tag{5}$$

$$n^2 = \frac{1}{a}\left[1 + \frac{3}{2}e^2 + \frac{3}{2}(2\sigma_1-\sigma_2) + \frac{3}{2}(2\sigma_3-\sigma_4) + \frac{2M_b r_c}{(r_c^2+T^2)^{3/2}}\right] \tag{6}$$

$$\sigma_1 = \frac{a^2-c^2}{5R^2},\ \sigma_2 = \frac{a^2-c^2}{5R^2},\ \sigma_3 = \frac{a'^2-c'^2}{5R^2},\ \sigma_4 = \frac{a'^2-c'^2}{5R^2},\ \mu = \frac{M_2}{M_1+M_2} \le \frac{1}{2},\ \sigma_i \ll 1,\ (i=1,2,3,4)$$

Where μ is the mass ratio, n the mean motion,M_1 denotes the mass of the larger primary, while M_2 stand for the mass of the smaller primary,q_i (i=1,2) are their radiation factors,r_1 and r_2 represent distances of the third body from the bigger and smaller primary respectively,$\sigma_1$and σ_2 denote the triaxiality of the bigger and smaller primary respectively, while σ_3 denote the triaxiality of the bigger primary and σ_4 stands for the triaxiality of the smaller primary.

The lengths of the axes are denoted by *a, b, c* for the bigger primary and a', b', c' for the smaller primary, a is the semi-major axis of the orbits of the primaries and e the enccentricity. M_b<<1 is the total mass of the belt, $r = \sqrt{\xi^2 + \eta^2}$ is the radial distance of the third body from the origin.$T = A + B,$ A and B are the parameters which determine the density profile of the belt ([14] Miyamoto and Nagai, 1975; [15] Jiang and Yeh, 2003; [16] Kushvah, 2008).The parameter B controls the size of the core of the density profile and is known as the core parameter,$r_c$is the radial distance of the third body from the collinear point under consideration.$W_2 = \frac{\mu(1-q_2)}{c_d}$ denotes the P-R drag of the smaller primary and C_d is the dimensionless speed of light. The configuration of the problem is shown below:

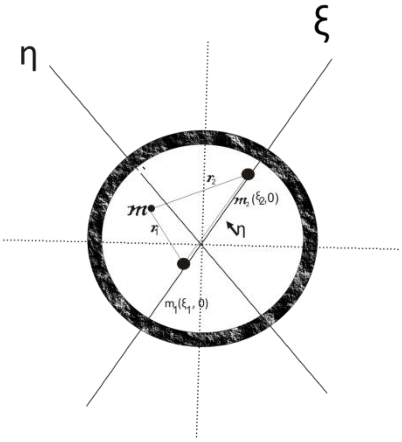


Fig. 2. The configuration of the problem

3. LOCATIONS AND STABILITY OF THE COLLINEAR LIBRATION POINTS

In order to obtain the location of collinear points we adopt the procedureused in [13]. At equilibrium points$\xi' = \eta' = \xi'' = \eta'' = \zeta' = \zeta'' = 0$.Therefore,equilibrium points lie in the ξη-plane and are the solutions of equations $\Omega_\xi = \Omega_\eta = 0, \zeta = 0$

The collinear points lieon the ξ- axis thereforeη = 0, using Equation (5) in above equations and avoiding situations $\xi = 1-\mu$ and$-\mu$ we have:

$$n^2\xi - \frac{(1-\mu)(\xi+\mu)q_1}{|\xi+\mu|^3} - \frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)q_1}{2|\xi+\mu|^5} - \frac{\mu(\xi+\mu-1)q_2}{|\xi+\mu-1|^3} - \frac{3\mu(\xi+\mu-1)(2\sigma_3-\sigma_4)q_2}{2|\xi+\mu-1|^5} - \frac{Mb\xi}{(r^2+T^2)^3/_2} = 0 \quad (7)$$

Now we consider,

$$\xi_2 - \xi_1 = 1, \xi_1 = -\mu, \xi_2 = 1-\mu \quad (8)$$

Rewiting equation (7) using equation (8) we obtain:

$$n^2\xi - \frac{(1-\mu)q_1}{|\xi-\xi_1|^2} - \frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2|\xi+\xi_1|^4} - \frac{\mu q_2}{|\xi-\xi_2|^2} - \frac{3\mu(2\sigma_3-\sigma_4)q_2}{2|\xi-\xi_2|^4} - \frac{Mb\xi}{(r^2+T^2)^3/_2} = 0 \quad (9)$$

In order to locate the collinear points $L_{1,2,3}$ we divide the orbital plane into three parts $\xi > \xi_2$, $\xi_1 < \xi < \xi_2$ and $\xi_1 > \xi$ with respect to the primaries.

3.1 Position of L_1 ($\xi > \xi_2$)

Let the collinear libration point L_1 be on the right hand side of the smaller primary at a distance ρ from it on the ξ- axis

In the interval ($\xi > \xi_2$) we let $\xi - \xi_2 = \rho$, $\xi - \xi_1 = 1 + \rho \Rightarrow \xi = 1 + \rho + \xi_1, \xi_2 - \xi_1 = 1$
$\Rightarrow \xi = 1 + \rho - \mu, r_1 = 1 + \rho, r_2 = \rho$ (10)

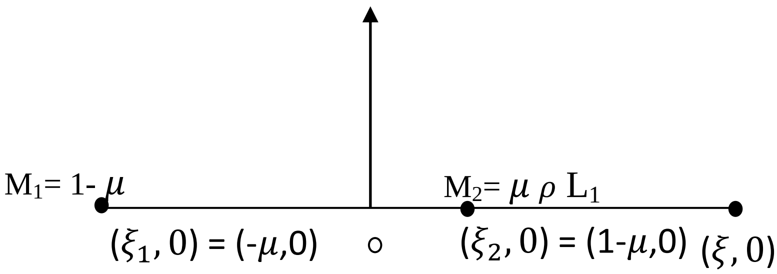


Fig. 3. Position of collinear libration point L_1

Thus by substituting equation (10) in the equation (9) we have:

$2n^2(1+\rho-\mu)(1+\rho)^4\rho^4q_1 - 2(1-\mu)(1+\rho)^2\rho^4q_1 - 3(1-\mu)(2\sigma_1-\sigma_2)\rho^4q_1 - 2\mu(1+$
$\rho4\rho2q2-3\mu2\sigma3-\sigma41+\rho4q2-Mb1+\rho-\mu\{(1+\rho-\mu)2+T2\}3\ 2$= 0 (11)

After expansion we obtain:

$2n^2\rho^9 + 2n^2(5-\mu)\rho^8 + 2n^2(2(5-2\mu))\rho^7 + 2(2n^2(5-3\mu) - (q_1 - \mu q_1 + \mu q_2))\rho^6 +$
$((2n^2(5-4\mu) - 4(q_1 - \mu q_1 + 2\mu q_2))\rho^5 + (2n^2(1-\mu) - 2(q_1 - \mu q_1 + 6\mu q_2) - 3(2\sigma_1 q_1 -$
$\sigma2q1+32\mu\sigma1q1-\mu\sigma2q1-32\mu\sigma3-\mu\sigma4q2\rho4-4\mu q2+6\mu\sigma3q2-3\mu\sigma4q2\rho3-2\mu q2+18\mu\sigma3q2-9\mu\sigma4q2$
$\rho2-38\mu\sigma3q2-4\mu\sigma4q2\rho-(3\mu2\sigma3-\mu\sigma4q2-Mb(1+\rho-\mu)\{(1+\rho-\mu)2+T2\}32$ = 0

(12)

3.2 Position of L_2 ($\xi_1 < \xi < \xi_2$)

Let the collinear libration point L_2 be on the left hand side of the smaller primary at a distance ρ from it on the ξ- axis

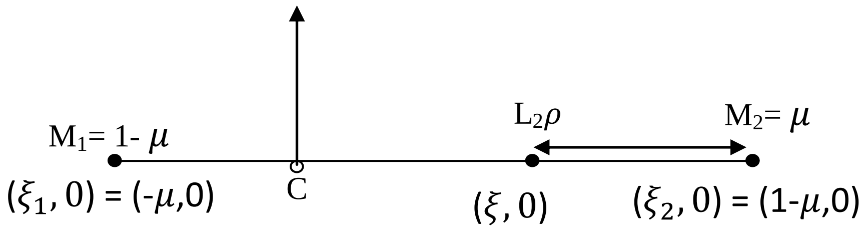


Fig. 4. Position of collinear libration point L_2

In the interval L_2 ($\xi_1 < \xi < \xi_2$), we let $\xi_2 - \xi = \rho; \xi - \xi_1 = 1 - \rho \Rightarrow \xi = 1 - \rho - \mu$ and $r_1 = 1 - \rho, r_2 = \rho$ (13)

Using equation (13) in the equation (9) we get:

$2n^2(1-\rho-\mu)(1+\rho)^4\rho^4 q_1 - 2(1-\mu)(1-\rho)^2\rho^4 q_1 - 3(1-\mu)(2\sigma_1 - \sigma_2)\rho^4 q_1 + 2\mu(1-$ $\rho 4\rho 2q2-3\mu 2\sigma 3-\sigma 41-\rho 4q2-Mb(1-\rho-\mu)\{(1+\rho-\mu)2+T2\}32 = 0$
(14)

Hence expanding equation (14) we obtain:

$-2n^2\rho^9 + 2n^2(5-\mu)\rho^8 - 2n^2(2(5-2\mu))\rho^7 + 2(2n^2(5-3\mu) - (q_1 - \mu q_1 - \mu q_2))\rho^6 +$ $(-2n^2(5-4\mu) + 4(q_1 - \mu q_1 - 2\mu q_2))\rho^5 + (2n^2(1-\mu) - 2(q_1 - \mu q_1 - 6\mu q_2) - 3(2\sigma_1 q_1 -$ $\sigma 2q1+32\mu\sigma 1q1-\mu\sigma 2q1+32\mu\sigma 3-\mu\sigma 4q2\rho 4-42\mu q2+6\mu\sigma 3q2-3\mu\sigma 4q2\rho 3+2\mu q2+18\mu\sigma 3q2-9\mu\sigma 4$ $q2\rho 2-38\mu\sigma 3q2-4\mu\sigma 4q2\rho+3\mu 2\sigma 3-\mu\sigma 4q2-Mb(1-\rho-\mu)\{(1+\rho-\mu)2+T2\}32 = 0$
(15)

3.3 Position of L_3 ($\xi_1 > \xi$)

Let the collinear libration point L_3 be on the left hand side of the bigger primary at a distance 1-ρ from it on the ξ - axis.

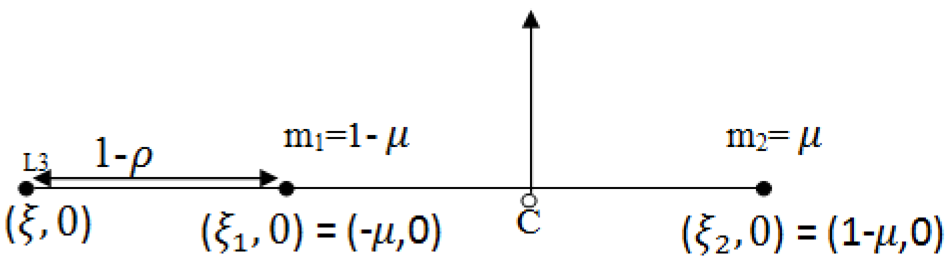


Fig. 5. Position of collinear libration point L_3

Finally, in the interval ($\xi_1 > \xi$) we let

$\xi_1 - \xi = 1 - \rho;\ \xi_2 - \xi = 2 - \rho$, and $r_1 = 1 - \rho;\ r_2 = 2 - \rho, \xi = -1 - \mu + \rho$ (16)

Using equation (16) in the equation (9) we get:

$$2n^2(\rho-1-\mu)(1-\rho)^4(2-\rho)^4q_1+2(1-\mu)(1-\rho)^2(2-\rho)^4q_1+3(1-\mu)(2\sigma_1-\sigma_2)(2-$$
$\rho4q1+2\mu1-\rho42-\rho2q2-3\mu2\sigma3-\sigma41-\rho4q2-Mb(-1-\mu+\rho)\{(-1-\mu+\rho)2+T2\}32 = 0$ (17)

Expanding equation (20) we get:

$$+2n^2\rho^9+2n^2(-13-\mu)\rho^8+2n^2(2(37+6\mu))\rho^7+2(-2n^2(121-31\mu)+(q_1-\mu q_1+\mu q_2))\rho^6+2\left(n^2(501-180\mu)-2(4q_1-5\mu q_1+4\mu q_2)\right)\rho^5+\left(-2n^2(681-321\mu)+2(40q_1-\right.$$
$41\mu q1-26\mu q2+32\sigma1q1-\sigma2q1-2\mu\sigma1q1+\mu\sigma2q1+2\mu\sigma3q2-\mu\sigma4q2)\rho4+2n2680+360\mu-882q1-2\mu q1+\mu q2+12(-4\sigma1q1+2\sigma2q1+4\mu\sigma1q1-2\mu\sigma1q1-2\mu\sigma3q2+\mu\sigma4q2\rho3+-2n2344+248\mu+826q1-26\mu q1-26\mu q1+18\sigma1q1-9\sigma2q1-18\mu\sigma1q1+9\mu\sigma2q1+241\mu q2+18\mu\sigma3q2-9\mu\sigma4q2)\rho2+2n2112+96\mu-8-16q1+16\mu q1-24\sigma1q1+12\sigma2q1+24\mu\sigma1q1-12\mu\sigma2q1-5\mu q2-3\mu\sigma3q2-12\mu\sigma4q2\rho1+2n2-16+16\mu+84q1-4\mu q1-6\sigma2q1-12\mu\sigma1q1+6\mu\sigma2q1+q2+32\mu\sigma3q2-\mu\sigma4q2$
$-Mb(-1-\mu+\rho)\{(-1-\mu+\rho)2+T2\}32= 0$
(18)

We shall further solve Eqs. (12), (15) and (18) numerically for the real values of ρ.Then using its values we shall find the positions of $L_{1,2,3.}$

3.4 Stability of the Collinear Equilibrium Points

We use the characteristic equation of the system as given by Singh and Isah (2021)13, to determine the stability of the collinear libration point L_i (i=1,2,3) which is:

$$\lambda^4-\left(\Omega^0{}_{\xi\xi}+\Omega^0{}_{\eta\eta}-4\right)\lambda^2+\Omega^0{}_{\xi\xi}\Omega^0{}_{\eta\eta}-\left(\Omega^0{}_{\xi\eta}\right)^2=0 \tag{19}$$

Taking the second partial derivative of equation (2), with$\eta=0$ we have:

$$\Omega^0{}_{\xi\xi}=(1-e^2)^{-1/2}\left[1+\frac{2}{n^2}\left\{\frac{(1-\mu)q_1}{|\xi+\mu|^3}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{|\xi+\mu|^5}+\frac{\mu q_2}{|\xi+\mu-1|^3}+\frac{3\mu(2\sigma_3-\sigma_4)q_2}{|\xi+\mu-1|^5}-\frac{Mb}{(r^2+T^2)^{3/2}}+\frac{Mb\xi^2}{(r^2+T^2)^{5/2}}\right\}\right]$$

$$\Omega^0{}_{\eta\eta}=(1-e^2)^{-1/2}\left[1-\frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{|\xi+\mu|^3}+\frac{3(1-\mu)(4\sigma_1-3\sigma_2)q_4}{2|\xi+\mu|^5}+\frac{\mu q_2}{|\xi+\mu-1|^3}+\frac{3\mu(4\sigma_3-3\sigma_4)q_1}{2|\xi+\mu-1|^5}+\frac{Mb}{(r_c{}^2+T^2)^{3/2}}\right\}-\right.$$
$4W2\eta\xi+\mu-1r23n$
(20)

$$\Omega^0{}_{\zeta\eta}=\Omega^0{}_{\eta\zeta}=0 \tag{21}$$

It is obvious that $\Omega^0{}_{\xi\xi}>0$ (22)

3.4.1 Stability of collinear point L_1 in the interval $\xi>\xi_2$)

In the first interval we have $\xi>\xi_2$)

$$r_1=(\xi+\mu)\Longrightarrow\xi=(r_1-\mu) \text{ and } r_2=(\xi+\mu-1) \tag{23}$$

Substituting equation (23) in the equation (7), we get:

$$\frac{(1-\mu)}{{r_1}^2}=n^2\xi-\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2{r_1}^4}-\frac{\mu q_2}{{r_2}^2}-\frac{3\mu(2\sigma_3-\sigma_4)q_1}{2{r_2}^4}+\frac{Mb(r_1-\mu)}{({r_c}^2+T^2)^{3}/_{2}} \quad (24)$$

Putting equation (24) in the second equation of (20), we obtain

$$\Omega^0{}_{\eta\eta}=(1-e^2)^{-1}/_{2}\left[1-\frac{1}{n^2}\left\{\frac{1}{r_1}\left(n^2\xi-\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2{r_1}^4}-\frac{\mu q_2}{{r_2}^2}-\frac{3\mu(2\sigma_3-\sigma_4)q_2}{2{r_2}^4}+\frac{Mb(r_1-\mu)}{({r_c}^2+T^2)^{3}/_{2}}\right)+\right.\right.$$

31−μ4σ1−3σ22r15+μq2r23+3μ4σ3−3σ4q22r25+Mbrc2+T232+4W2ηξ+μ−1r23n
(25)

Neglecting higher order terms in e^2,a,σ_i (i=1,2,3,4) and M_b we have

$$\Omega^0{}_{\eta\eta}=\frac{\mu}{r_1}+\frac{1}{n^2}\left\{\frac{\mu q_2}{{r_2}^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{3\mu q_2}{2{r_2}^4}\left\{\frac{(4\sigma_3-3\sigma_4)}{r_2}-\frac{(2\sigma_3-\sigma_4)}{r_1}\right\}-\frac{3(1-\mu)q_1}{{r_1}^5}(\sigma_1-\sigma_2)\right.$$
$$\left.+\frac{M_b(1+r_1-\mu)}{({r_c}^2+T^2)\,{}^{3}/_{2}}\right\}-\frac{4W_2\eta(\xi+\mu-1)}{{r_2}^3n}$$

Thus $\Omega^0{}_{\eta\eta}<0$, since $\mu<\frac{1}{2}, \sigma_1(i=1,2,3,4)\ll 1, r_1>1, r_2<1$ and Mb<<1

Also, for the collinear points lying in the interval, $\xi_1<\xi<\xi_2$ and $\xi_1>\xi$ with respect to their primaries with η=ζ=0, we have $\Omega^0{}_{\xi\xi}>0, \Omega^0{}_{\eta\eta}<0, \Omega^0{}_{\zeta\eta}=0$.

Since $\Omega^0{}_{\xi\xi}\Omega^0{}_{\eta\eta}-(\Omega^0{}_{\xi\eta})^2$<0, the discriminant of equation (19) is positive for all intervals and therefore the characteristic roots can be written as;

$$\lambda_{1,2}=\pm c \text{ and } \lambda_{3,4}=\pm id \quad (26)$$

Where c and d are real numbers

We therefore conclude that the collinear libration points are unstable due to the nature of the characteristic root of equation (26).They are mixtures of real and complex roots, for the collinear libration points to be stable the four roots must pure be imaginary roots [17].

4. NUMERICAL APPLICATION

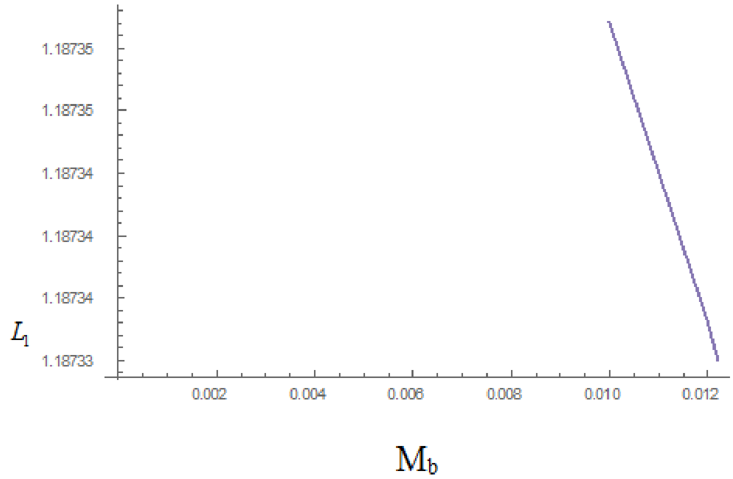
In this section the location of collinear libration points L_i (i = 1,2,3) and their corresp[illegible]ng characteristic roots for the two binary syst[illegible] FL virginis and Procyon are obtained numerically.Using the definition of parametres as in [13] which we restate here,we have the radiation pressure factors q_1 and q_2 of the bigger and the smaller primary calculated as: q=1-(^ × L/r ρ M) on the basis of Stefan boltzmann'slaw, q is the radiation pressure efficiency of a star, where M mass of a star and and L the luminousity of a star, r is the radius and ρ the density of a moving test particle, ^=($\frac{3}{16\pi CG}$) is a constant. In the centimeter –gram second system of units(C.G.S system) ^= 2.9838 X 10^{-5} .we take r = 2×10^{-2}cm and ρ = 1.4gcm^{-3} for some dust particle. Table 1 contains the numerical data of the binary system. Table 2 shows the numerical computation of the Location of collinear equilibrium points $L_{1,2,3}$ of the two binary systems (FL virginis and Procyon) for different values of the belt and PR-drag. The characteristic roots are presented in Tables 3 and 4.

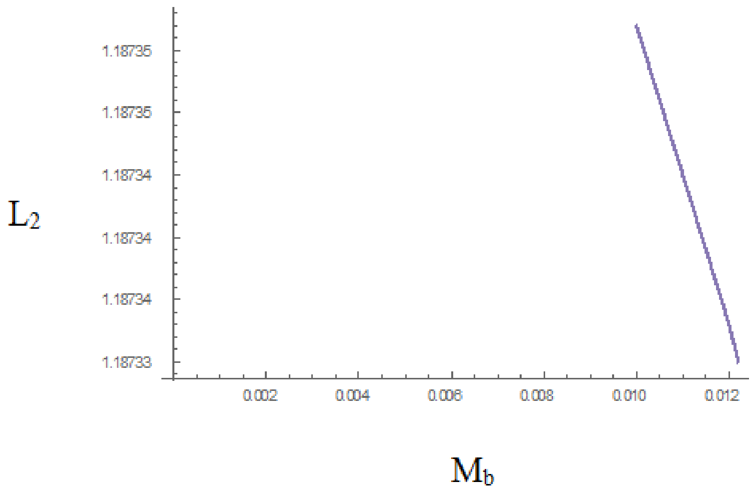
The effects of the gravitational pontetial from the belt and Pr-drag on the positions of equilibrium points are shown on the graphs in Figs. 6 and 7. The numerical data for the binary system (Table 1) is obtained from [13].

5. DISCUSSION AND CONCLUSION

In Tables 2-4 we have used equation (12), (15) and (18) to compute numerically, using the software mathematica, the positions of the collinear equilibrium points for the different values of the beltand the pr-drag of the two binaries (FL virginis and Procyon). We can see from Table 2 and Table 3that the dynamical effect of the belt and pr-drag on the positions of the collinear equilibrium points of FL virginis and procyon are similar because in both cases the collinear points $L_{1,2,3}$ move in the same direction. Both L_1 and L_2moves towards the smaller primary , while L_3 shifts towards the bigger primary. We present in Table 4 and Table 5 the corresponging roots to the locations in Table 2 and Table 3.

The stability of collinear equilibrium points are obtained by substituting Equations (20) and (21) in (19). The characteristic roots obtained are shown in Table 4 and 5 for the systems FL virginis and Procyon. The characteristic roots obtained in Table 4 and 5 using the eccentricity, semi-major axis and radiation factor of the two binary systems with arbitrarily chosentriaxiaity coefficients,belt and Pr-drag areunstable due to the absence of pure imaginary roots or complex roots with negative real parts in the four roots ((λ_i (i=1,2,3,4)).The presence of positive real parts in the roots shows that the collinear libration points are unstable.This instability behaviour was confirmed by [4] Singh and Umar,; [18] Kumar and Naraya, 2012; [19] Singh and Tokyaa, 2017). We have also shown graphically, the effect of the belt and Pr-drag on the positions of collinear points (Figs. 6 and 7) by substutimg the collinear points of the binaries and the increasingly varied values of the belt and Pr-drag into equation (19).The graphs show clearly that the position of collinear points move uniformly with increasing values of the belt and Pr-drag.





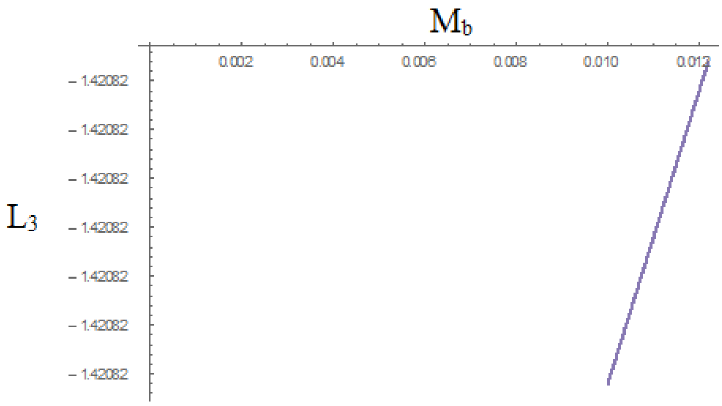


Fig. 6. The effect of pontetial from the belt and Pr-Drag on $L_{1,2,3}$ of FL virginis

Table 1. Numerical data for the binary system

Binary system	Masses $M_\odot$		*e*	*a*	Luminosity $L_\odot$		Spectral type
	M_1	M_2			$L_{1\odot}$	$L_{2\odot}$	
FL virginis	0.076	0.067	0.3	0.9306''	1.372×10^{-3}	1.078×10^{-3}	M5se/M7
Procyon	1.53	0.617	0.4	4.3''	6.93	0.00049	F5/DA

Sorce: Stellar-DataBase/The American Astronomical Society/Wikepedia

Table 2. Locations of collinear equilibrium points L_i(i = 1,2,3) for the binary system FL virginis and procyon for different values of the belt

Binary system	Parameters values				Location		
FL virginis	M_b	μ	e	a	L_1	L_2	L_3
	0.001	04685	0.3	0.219	1.173789	0.33456876	-1.5326169
	0.002				1.173779	0.33456868	-1.5326161
	0.003				1.173768	0.33456861	-1.5326152
	0.004				1.173754	0.33456855	-1.5326148
Procyon	0.001	0.2874	0.4	0.714	1.361245	0.19976511	-1.4634387
	0.002				1.361239	0.19976501	-1.4634379
	0.003				1.361232	0.19976431	-1.4634371
	0.004				1.361227	0.19976422	-1.4634364

For FL virginisq_1= 0,9996, q_2= 0,9995, W_2=3.39× 10^{-10}
ForProcyonq_1= 0,9995, q_2= 0,9994, W_2=3.39× 10^{-10}

Table 3. Locations of collinear equilibrium points L_i(i = 1,2,3) for the binary system FL virginis and procyon for different values of the PR-drag

Binary system	Parameters values				Location		
FL virginis	W_2	μ	e	a	L_1	L_2	L_3
	1.39×10^{-10},	04685	0.3	0.219	1.244739	0.578634	-1.4849688
	2.39×10^{-10},				1.244731	0.578624	-1.4849678
	3.39×10^{-10},				1.244726	0.578616	-1.4849667
	4.39×10^{-10},				1.244720	0.578607	-1.4849658
Procyon	1.39×10^{-10},	0.2874	0.4	0.714	1.128555	0.449471	-1.2433861
	2.39×10^{-10},				1.128545	0.449463	-1.2433850
	3.39×10^{-10},				1.128537	0.449455	-1.2433841
	4.39×10^{-10},				1.128529	0.449447	-1.2433832

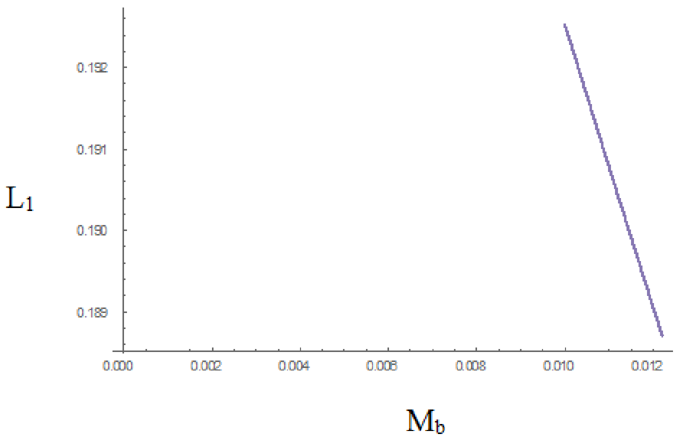
For FL virginis q_1= 0,9996, q_2= 0,9995, M_b=0.01
For Procyon q_1= 0,9995, q_2= 0,9994, M_b=0.01

Table 4. The characteristic roots ($\lambda_{1,2}$, $\lambda_{3,4}$) of collinear points for the system FL virginis for different values of the belt

L_1	$\lambda_{1,2}$	$\lambda_{3,4}$	Stability behaviour
1.173789	$\pm$ 0.652854	$\pm 1.26644i$	Unstable
1.173779	$\pm$ 0.652819	$\pm 1.26634i$	Unstable
1.173768	$\pm$ 0.652792	$\pm 1.26625i$	"
1.173754	$\pm$ 0.652740	$\pm 1.26613i$	"
L_2	$\lambda_{1,2}$	$\lambda_{3,4}$	
0.33456876	$\pm$(5919.01-072.62 i)	$\pm$(5919.01+6072.62 i)	Unstable
0.33456868	$\pm$(15670.3-7545.3 i)	$\pm$(17545.3+1567.3i)	"
0.33456861	$\pm$(17810.2-0147.9 i)	$\pm$(17810.2+20147.9 i)	"
0.33456855	$\pm$(20410.4-3337.2 i)	$\pm$(20410.4+23337.2 i)	"
L_3	$\lambda_{1,2}$	$\lambda_{3,4}$	
-1.5326169	$\pm$(66.9213-64.873i)	$\pm$(54.9571+47.887i)	Unstable
-1.5326161	$\pm$(66.8540-57.873i)	$\pm$(54.626+46.926i)	"
-1.5326152	$\pm$(66.8101-52.873i)	$\pm$(54.417+43.9029i)	"
-1.5326148	$\pm$(66.9213-48.873i)	$\pm$(54.309+57.7923i)	"

Table 5. The characteristic roots ($\lambda_{1,2}$, $\lambda_{3,4}$) of collinear points for the system procyon for different values of the belt

L_1	$\lambda_{1,2}$	$\lambda_{3,4}$	Stability behaviour
1.361245	$\pm$0.745982	$\pm$1.4558i	Unstable
1.361239	$\pm$0.746113	$\pm 1.4543i$	"
1.361232	$\pm$0.746153	$\pm$1.45401i	"
1.361227	$\pm$0.746208	$\pm$1.4535i	"
L_2	$\lambda_{1,2}$	$\lambda_{3,4}$	
0.19976511	$\pm$(0.587461-0.675612i)	$\pm$(0.578865+0.698567i)	Unstable
0.19976501	$\pm$(0.566124-0.672713i)	$\pm$(0.567891+0.687944i)	"
0.19976431	$\pm$(0.53659-0.665445i)	$\pm$(0.557875+0.685432i)	"
0.19976422	$\pm$(0.506454-0.660482i)	$\pm$(0.53475+0.683345i)	"
L_3	$\lambda_{1,2}$	$\lambda_{3,4}$	
-1.4634387	$\pm$(0.56789-1.47111i)	$\pm$0.454078+1.43211i)	Unstable
-1.4634379	$\pm$(342.444-298.197i)	$\pm$(356.292+288.213i)	"
-1.4634371	$\pm$(320.21-275.698i)	$\pm$(347.34+265.843i)	"
-1.4634364	$\pm$(310.098-222.143i)	$\pm$(326.174+277.197i)	"



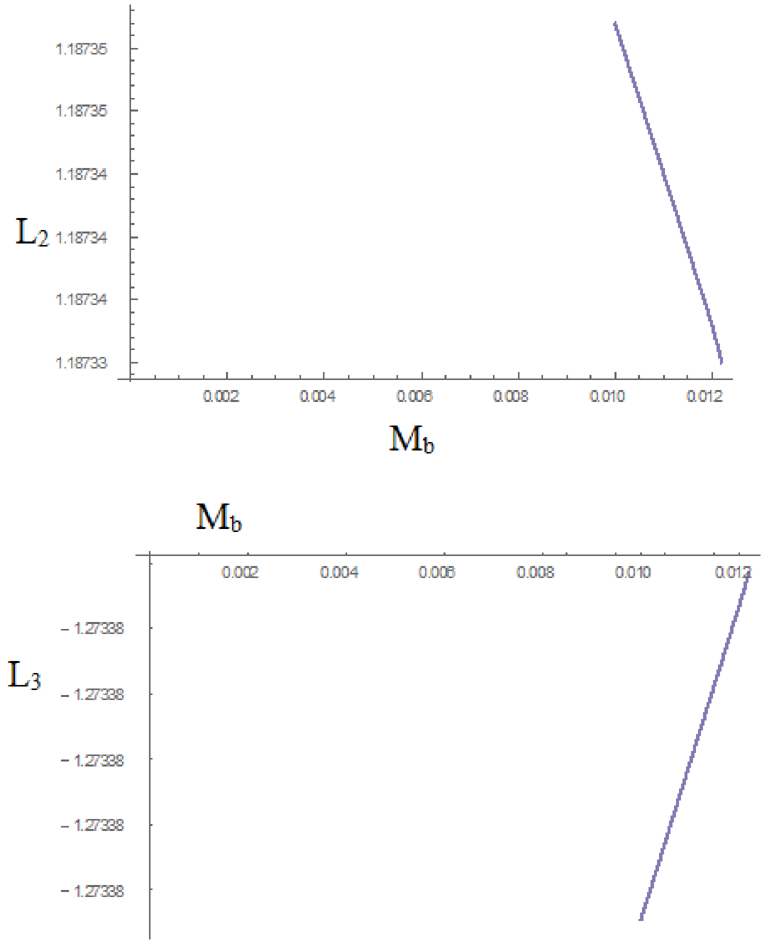


Fig. 7. The effect of pontetial from the Belt and PR-Drag on $L_{1,2,3}$ of FL procyon

We have studied the positions and stability of collinear equilibrium points in the elliptic restricted three body problem under the influence of triaxiality, radiation, Pr-drag and gravitational potential from the belt. We found that the positions and linear stability of the collinear equilibrium points are affected significantly by triaxiality, radiation, Pr-drag and gravitational potential from the belt. The collinear equilibrium points are found to be unstable.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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