On the Algebraic Properties of Quasi-affine Bijective Transformations

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Authors' contributions

This work was carried out in collaboration between both authors. Author CMK designed the study and wrote the first draft of the manuscript. Authors AML and CMK managed the analyses of the study. Author CMK managed the literature searches. Both authors read and approved the final manuscript.

Original Research Article

Abstract

A quasi-affine transformation, being the whole part of a rational affine transformation, is the discretized form of an affine transformation. Introduced by Marie-André Jacob-Da Col, it has been the subject of numerous studies. This article is devoted to the study of the algebraic structures of some quasi-affine bijective transformations, in particular the discrete translations of isolated points and Pythagorean rotations.

1 Introduction

A quasi-affine transformation is the correspondent of the affine transformation in discrete space. This is the discretization of the affine transformation. Often used in image processing applications, several articles have focused on the study of quasi-affine transformation in the discrete plane [1,2,3]. Precisely on the link of

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quasi-affine transformation with tiling, on the dynamic system associated with the QAT and on the various algorithms linked to QAT.

Blot and Coeurjolly [4] resumed the study of quasi-affine mappings in a higher dimensional discrete space. Pluta and Romon [5] approached the notion of discrete isometries digitized of subsets of planes. A comparative study has shown that some properties of affine transformations are not conserved by discrete affine transformations (QAT) [6].

In section 2, we first detail basic terminology and preliminaries and we prove in general the algebraic structure of bijective quasi-affine transformation. Then, Section 3 contains the main results with the algebraic structure for discrete translations of isolated points and Pythagorean rotations in $\mathbb{Z}^2$

2 Basic Terminology and Preliminaries

In this article, we will use the following concepts and notations [7, 1]:

- If x is a real, $\lfloor x \rfloor$ represents the entire part of x which is the largest integer less than and equal to x.
- If x and y are two integers with $y \neq 0$, $\left\lfloor \frac{x}{y} \right\rfloor$ respectively represents the integer quotient of x by y.
- If x and y are integers, then *gcd(x,y)* is the greatest common divisor of x and y.

2.1 Definition

A quasi-affine transformation $\lfloor f \rfloor$ (QAT) is a discrete transformation defined such as:

$$\lfloor f \rfloor \; : \; \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{cases} x' = \left\lfloor \dfrac{ax + by + e}{\omega} \right\rfloor \\ y' = \left\lfloor \dfrac{cx + dy + f}{\omega} \right\rfloor \end{cases}$$

where $a, b, c, d, e, f \, \omega \in \mathbb{Z}$ with $\omega > 0$. It is the integer part of a rational affine transformation [2, 3, 8, 9].

2.2 Definition

A quasi-linear transformation $\lfloor u \rfloor$ (QLT) is a discrete linear transformation such as:

$$\lfloor u \rfloor \; : \; \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{cases} x' = \left\lfloor \dfrac{ax + by}{\omega} \right\rfloor \\ y' = \left\lfloor \dfrac{cx + dy}{\omega} \right\rfloor \end{cases}$$

where $a, b, c, d \, \omega \in \mathbb{Z}$ with $\omega > 0$. It is the integer part of a rational linear transformation. The matrix $\mathcal{A} = \frac{1}{\omega} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called the quasi-linear transformation matrix [4].

2.3 Example

Assume a quasi-affine transformation
$$\lfloor f \rfloor : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{cases} x' = \left\lfloor \dfrac{x - 2y - 3}{2} \right\rfloor \\ y' = \left\lfloor \dfrac{2x + y + 4}{2} \right\rfloor \end{cases}$$

Its graphic representation in a subset of $\mathbb{Z}^2$ $D = [\![-5; 5]\!] \times [\![-5; 5]\!]$ is given by the Fig. 1 below:
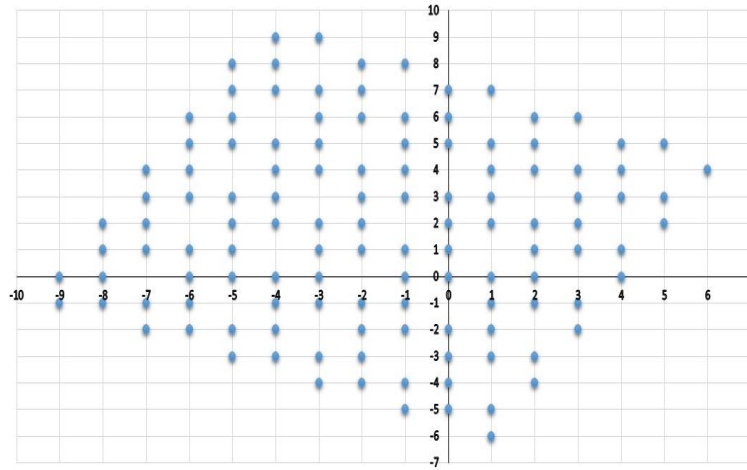

Fig. 1. The image of space $D \subset \mathbb{Z}^2$ by the transformation $\lfloor f \rfloor$ with $\omega = 2$

2.4 Definition

The QAT is called contracting if $\omega^2 > |\det(A)|$, otherwise it is called dilating [4, 10].

We note that the quasi-affine transformation defined above is dilating (see Fig. 1) while that defined by

$$\lfloor f' \rfloor : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = \left\lfloor \frac{x - 2y - 3}{4} \right\rfloor \\ y' = \left\lfloor \frac{2x + y + 4}{4} \right\rfloor \end{cases} \quad \text{is contracting.}$$
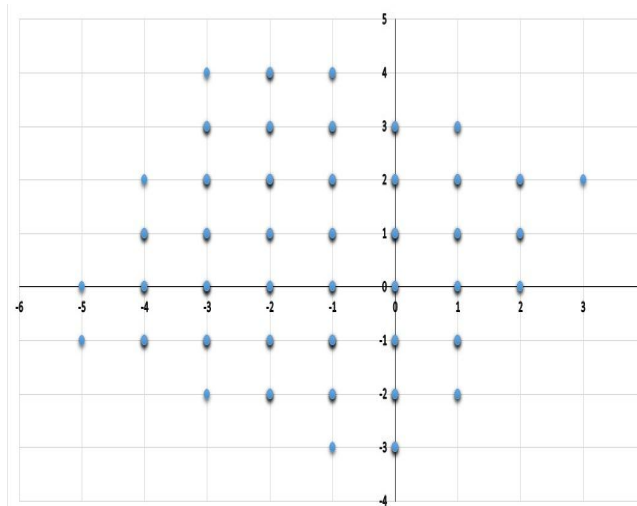
The Fig. 2 is a representation of the QAT contracting

Fig. 2. The image of the subspace $D \subset \mathbb{Z}^2$ by the transformation $\lfloor f' \rfloor$, if $\omega = 4$

2.5 Definition

Let $\lfloor f \rfloor$ be a QAT contracting. For $(x', y') \in \mathbb{Z}^2$, we denote
$$P_{(x', y')} = \{(x, y) \in \mathbb{Z}^2 / \lfloor f \rfloor (x, y) = (x', y')\} = f^{-1}(x', y')$$

$P_{(x', y')}$ is called order 1 paving of index (x',y') of $\lfloor f \rfloor$.

2.6 Example

If we consider the previous example for a QAT contracting $\lfloor f' \rfloor$, the different order 1 paving are:

$P_{0,0} = \{(-1, -2), (0, -2), (0, -3)\}$
$P_{0,1} = \{(1, -1), (1, -2), (2, -1), (2, -2)\}$
$P_{1,0} = \{(0, -4), (1, -4), (1, -3)\}$
$P_{1,1} = \{(2, -4), (2, -3), (3, -3)\}$
$P_{0,2} = \{(3, -1), (3,0), (4, -1)\}$

2.7 Theorem

The quasi-linear application $\lfloor u \rfloor$ (QLT) is bijective if $\omega = \gcd(|a|, |b|, |c|, |d|)$ and $\det(A) \neq 0$

Proof: In fact, consider $0 < \omega \in \mathbb{Z}$ such that $\omega = \gcd(|a|, |b|, |c|, |d|)$. Prove that $\lfloor u \rfloor$ (QLT) is injective and surjective.

(1) Injectivity. For all $(x, y), (x', y') \in \mathbb{Z}^2$, we have $\lfloor u \rfloor (x; y) = \lfloor u \rfloor (x', y')$ then

$$\begin{cases} \left\lfloor \frac{ax+by}{\omega} \right\rfloor = \left\lfloor \frac{ax'+by'}{\omega} \right\rfloor \\ \left\lfloor \frac{cx+yd}{\omega} \right\rfloor = \left\lfloor \frac{ax'+by'}{\omega} \right\rfloor \end{cases} \implies \begin{cases} \left\lfloor \frac{a}{\omega}x + \frac{b}{\omega}y \right\rfloor = \left\lfloor \frac{a}{\omega}x' + \frac{b}{\omega}y' \right\rfloor \\ \left\lfloor \frac{c}{\omega}x + \frac{d}{\omega}y \right\rfloor = \left\lfloor \frac{c}{\omega}x' + \frac{d}{\omega}y' \right\rfloor \end{cases}$$
As $\omega = \gcd(a, b, c, d)$, the ratios $\frac{a}{\omega}, \frac{b}{\omega}, \frac{c}{\omega}, \frac{d}{\omega} \in \mathbb{Z}$.

Thus : $\begin{cases} \frac{a}{\omega}(x - x') + \frac{b}{\omega}(y - y') = 0 \\ \frac{c}{\omega}(x - x') + \frac{d}{\omega}(y - y') = 0 \end{cases}$ (1)

Knowing than $\det(A) \neq 0$ and $\omega^2 \neq 0$, the determinant of the system (1) is $\omega^2 \det(A) \neq 0$, then the system of linear equations possesses a solution unique[11]. The solution of the equation leads to $x = x'$ and $y = y'$. Hence, $(x, y) = (x', y')$ and $\lfloor u \rfloor$ is injective.

(2) Surjectivity. For all $(x', y') \in \mathbb{Z}^2, \exists (x, y) \in \mathbb{Z}^2$ such that $\lfloor u \rfloor (x, y) = (x', y')$, we have:

$$\begin{cases} \left\lfloor \frac{Ax+by}{\omega} \right\rfloor = x' \\ \left\lfloor \frac{cx+dy}{\omega} \right\rfloor = y' \end{cases} \implies \begin{cases} \left\lfloor \frac{ax}{\omega} + \frac{by}{\omega} \right\rfloor = x' \\ \left\lfloor \frac{cx}{\omega} + \frac{dy}{\omega} \right\rfloor = y' \end{cases} \implies \begin{cases} \frac{a}{\omega}x + \frac{b}{\omega}y = x' \\ \frac{c}{\omega}x + \frac{d}{\omega}y = y' \end{cases} \implies \begin{cases} ax + by = \omega x' \\ cx + dy = \omega y' \end{cases}$$

By solving linear systems of diophantine equations, we obtain the following solution:

$$(x, y) = \left(\frac{\omega(dx' - by')}{\det(A)}, \frac{\omega(ay' - cx')}{\det(A)} \right) \in \mathbb{Z}^2$$

Then $\lfloor u \rfloor$ is surjective. Hence, the quasi-linear transformation $\lfloor u \rfloor$ with $\omega = \gcd(|a|, |b|, |c|, |d|))$ is bijective.
$\square$

2.8 Theorem

The set of quasi-linear bijective transformations of $\mathbb{Z}^2$ with the usual composition has a group structure.

Proof. In fact,

(1) **Stability**: Let $\lfloor u \rfloor$ and $\lfloor u' \rfloor$ two quasi-linear transformations defined in $\mathbb{Z}^2$ respectively by $\lfloor u \rfloor (x, y) = (x', y')$ with $\begin{cases} x' = \left\lfloor \frac{ax+by}{\omega} \right\rfloor \\ y' = \left\lfloor \frac{cx+dy}{\omega} \right\rfloor \end{cases}$ and by $\lfloor u' \rfloor (x, y) = (x', y')$ with $\begin{cases} x' = \left\lfloor \frac{a'x+b'y}{\omega'} \right\rfloor \\ y' = \left\lfloor \frac{c'x+d'y}{\omega'} \right\rfloor \end{cases}$. We defined the homogenous matrix of $\lfloor u \rfloor$ by $M_{\lfloor u \rfloor} = \frac{1}{\omega}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that of $\lfloor u' \rfloor$ by $M_{\lfloor u' \rfloor} = \frac{1}{\omega}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.The matrix of $\lfloor u' \rfloor \circ \lfloor u \rfloor$ is defined by

$$A_{\lfloor u' \rfloor} \times A_{\lfloor u \rfloor} = \frac{1}{\omega\omega'}\begin{pmatrix} a'a + bc' & a'b + db' \\ c'a + d'c & c'b + d'b \end{pmatrix}$$

So $\lfloor u' \rfloor \circ \lfloor u \rfloor$ is the quasi-linear bijective transformation defined by $\lfloor u' \rfloor \circ \lfloor u \rfloor (x, y) = (x', y')$ with $\begin{cases} x' = \left\lfloor \frac{(a'a+bc')x+(a'b+db')y}{\omega\omega'} \right\rfloor \\ y' = \left\lfloor \frac{(c'a+d'c)x+(c'b+d'b)y}{\omega\omega'} \right\rfloor \end{cases}$

(2) **Associativity:** The multiplication of the matrix is associative; we deduce that the composition of bijective QLT is associative. Hence

$[u] \circ ([u'] \circ [u'']) = ([u] \circ [u']) \circ [u'']$
.

(3) **The existence of the neutral element.** The transformation id defined by $\lfloor id \rfloor (x, y) = \left(\left\lfloor \frac{x}{\omega} \right\rfloor, \left\lfloor \frac{y}{\omega} \right\rfloor \right)$ with $\omega = 1$,is the neutral element of the group of the quasi-linear transformations with the law of composition.

Then the set of quasi-linear bijective transformations of $\mathbb{Z}^2$ with the usual composition has a group structure.$\square$

(4) **Symmetrization.** A quasi-linear bijective transformation

$\lfloor u \rfloor : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = \left\lfloor \frac{ax + by}{\omega} \right\rfloor \\ y' = \left\lfloor \frac{cx + dy}{\omega} \right\rfloor \end{cases}$$

admit for inverse

$\lfloor u \rfloor^{-1} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = \left\lfloor \frac{\omega dx + \omega by}{\det A} \right\rfloor \\ y' = \left\lfloor \frac{-\omega cx + \omega ay}{\det A} \right\rfloor \end{cases} \qquad \square$$

2.9 Theorem

The quasi-affine transformation $\lfloor f \rfloor$(QAT) is bijective, if $\omega = \gcd(|a|, |b|, |c|, |d|, |e|, |f|)$ and $\det(A) \neq 0$.

Proof: Let $\omega = \gcd(|a|, |b|, |c|, |d|, |e|, |f|)$, and suppose $0 < \omega \in \mathbb{Z}$. Let us show that $\lfloor f \rfloor$(QAT) is injective and surjective.

(1) Injectivity. For all $(x, y), (x', y') \in \mathbb{Z}^2$, we have $\lfloor f \rfloor(x, y) = \lfloor f \rfloor(x', y')$ then

$$\begin{cases} \left\lfloor \dfrac{ax + by + e}{\omega} \right\rfloor = \left\lfloor \dfrac{ax' + by' + e}{\omega} \right\rfloor \\ \left\lfloor \dfrac{cx + yd + f}{\omega} \right\rfloor = \left\lfloor \dfrac{ax' + by' + e}{\omega} \right\rfloor \end{cases} \Rightarrow \begin{cases} \left\lfloor \dfrac{a}{\omega}x + \dfrac{b}{\omega}y + \dfrac{e}{\omega} \right\rfloor = \left\lfloor \dfrac{a}{\omega}x' + \dfrac{b}{\omega}y' + \dfrac{e}{\omega} \right\rfloor \\ \left\lfloor \dfrac{c}{\omega}x + \dfrac{d}{\omega}y + \dfrac{f}{\omega} \right\rfloor = \left\lfloor \dfrac{c}{\omega}x' + \dfrac{d}{\omega}y' + \dfrac{f}{\omega} \right\rfloor \end{cases}$$

As $\omega = \gcd(|a|, |b|, |c|, |d|, |e|, |f|)$, the ratios $\frac{a}{\omega}, \frac{b}{\omega}, \frac{c}{\omega}, \frac{d}{\omega}, \frac{e}{\omega}, \frac{f}{\omega} \in \mathbb{Z}$.

Hence : $\begin{cases} \dfrac{a}{\omega}(x - x') + \dfrac{b}{\omega}(y - y') = 0 \\ \dfrac{c}{\omega}(x - x') + \dfrac{d}{\omega}(y - y') = 0 \end{cases}$. (1)

Knowing than $\det(A) \neq 0$ and $\omega^2 \neq 0$, the determinant of system (1) is $\frac{\det(A)}{\omega^2} \neq 0$, then the system of linear equations possesses a unique solution [11]. The solution of the equation leads to $x = x'$ and $y = y'$. Hence, $(x, y) = (x', y')$ and $\lfloor f \rfloor$ is injective.

(2) Surjectivity. For all $(x', y') \in \mathbb{Z}^2$, $\exists (x, y) \in \mathbb{Z}^2$ such that $\lfloor f \rfloor(x, y) = (x', y')$, we have:

$$\begin{cases} \left\lfloor \dfrac{ax + by + e}{\omega} \right\rfloor = x' \\ \left\lfloor \dfrac{cx + dy + f}{\omega} \right\rfloor = y' \end{cases} \Rightarrow \begin{cases} \left\lfloor \dfrac{a}{\omega}x + \dfrac{b}{\omega}y + \dfrac{e}{\omega} \right\rfloor = x' \\ \left\lfloor \dfrac{c}{\omega}x + \dfrac{d}{\omega}y + \dfrac{f}{\omega} \right\rfloor = y' \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{a}{\omega}x + \dfrac{b}{\omega}y = x' - \dfrac{e}{\omega} \\ \dfrac{c}{\omega}x + \dfrac{d}{\omega}y = y' - \dfrac{f}{\omega} \end{cases} \Rightarrow \begin{cases} ax + by = \omega x' - e \\ cx + dy = \omega y' - f \end{cases}$$

By solving linear systems of diophantine equations, we obtain the following solution:

$$(x, y) = \left(\frac{\omega(dx' - by') + bf - de}{\det(A)}, \frac{\omega(ay' - cx') + ce - af}{\det(A)} \right) \in \mathbb{Z}^2$$

Then $\lfloor f \rfloor$ is surjective. Hence quasi-affine transformation $\lfloor f \rfloor$ such that $\det(A) \neq 0$ and $\omega = \gcd(|a|, |b|, |c|, |d|, |e|, |f|)$ is bijective.$\square$

Remarks

1. The quasi-affine bijective transformation is dilating.
2. The tiling of the quasi-affine bijective transformation corresponds to the point image by the inverse of the bijective quasi-affine application.

2.10 Theorem

The set of quasi-affine bijective transformations of $\mathbb{Z}^2$ with the usual composition has a group structure.

Proof: In fact,

(1) **Stability**: Let $\lfloor f \rfloor$ and $\lfloor g \rfloor$ be two quasi-affine applications defined in $\mathbb{Z}^2$ respectively by

$$\lfloor f \rfloor(x, y) = (x', y') \text{ with } \begin{cases} x' = \left\lfloor \frac{ax+by+e}{\omega} \right\rfloor \\ y' = \left\lfloor \frac{cx+dy+f}{\omega} \right\rfloor \end{cases} \text{ and by}$$

$$\lfloor g \rfloor(x, y) = (x', y') \text{ with } \begin{cases} x' = \left\lfloor \frac{a'x+b'y+e'}{\omega'} \right\rfloor \\ y' = \left\lfloor \frac{c'x+d'y+f'}{\omega'} \right\rfloor \end{cases}$$

We define the homogeneous matrix of $\lfloor f \rfloor$ by

$$M_{\lfloor f \rfloor} = \frac{1}{\omega} \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \text{ and of } \lfloor g \rfloor \text{ by}$$

$$M_{\lfloor g \rfloor} = \frac{1}{\omega'} \begin{pmatrix} a' & b' & e' \\ c' & d' & f' \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix of $\lfloor g \rfloor \circ \lfloor f \rfloor$ is defined by

$$M_{\lfloor g \rfloor \circ \lfloor f \rfloor} = M_{\lfloor g \rfloor} \times M_{\lfloor f \rfloor}$$
$$= \frac{1}{\omega\omega'} \begin{pmatrix} a'a + b'c & a'b + db' & a'e + b'f + e' \\ c'a + d'c & c'b + d'd & c'e + d'f + f' \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, $\lfloor g \rfloor \circ \lfloor f \rfloor$ is a quasi-affine bijective transformation defined by $\lfloor g \rfloor \circ \lfloor f \rfloor (x', y') = (x', y')$ with
$$\begin{cases} x' = \left\lfloor \frac{(a'a+b'c)x+(a'b+db')y+a'e+b'f+e'}{\omega\omega'} \right\rfloor \\ y' = \left\lfloor \frac{(c'a+d'c)x+(c'b+d'd)y+c'e+d'f+f'}{\omega\omega'} \right\rfloor \end{cases}$$

(2) **Associativity:** The multiplication of matrix is associative; we deduce that the usually composition of bijective QAT is associative. Hence,

$$([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f])$$

(3) **The neutral element existence.** The transformation $\lfloor id \rfloor$ defined by $\lfloor id \rfloor(x, y) = \left(\left\lfloor \frac{x}{\omega} \right\rfloor, \left\lfloor \frac{y}{\omega} \right\rfloor \right)$ with $\omega = 1$, is the neutral element of the group of the quasi-affine transformations with the usually composition.

(4) **Symmetrization.** A quasi-affine transformation $\lfloor f \rfloor$ admit for inverse is defined by

$$\lfloor f \rfloor^{-1} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = \left\lfloor \frac{\omega(dx - by) + bf - de}{\det(A)} \right\rfloor \\ y' = \left\lfloor \frac{\omega(ay - cx) + ce - af}{\det(A)} \right\rfloor \end{cases}$$

with $\det(A) \neq 0$

Hence, the set of quasi-affine bijective transformations of $\mathbb{Z}^2$ with the usual composition has a group structure.☐

3 The Algebraic Structure of Quasi-Affine Transformations

In this section, we study the usually Algebraic structures of some quasi-affine transformations.

3.1 Discrete translations

3.1.1 Definition

A discrete translation $\boldsymbol{T}_u$ of the vector u is defined by [2]

$$\mathbf{T}_u: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = x + \left\lfloor u + \frac{1}{2} \right\rfloor \\ y' = y + \left\lfloor v + \frac{1}{2} \right\rfloor \end{cases}$$

3.1.2 Example

Let $u = (2, -4)$ a vector translation, $\mathbf{T}_u$ is defined by:

$$\mathbf{T}_u : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = x + \left\lfloor 2 + \frac{1}{2} \right\rfloor \\ y' = y + \left\lfloor -4 + \frac{1}{2} \right\rfloor \end{cases}$$

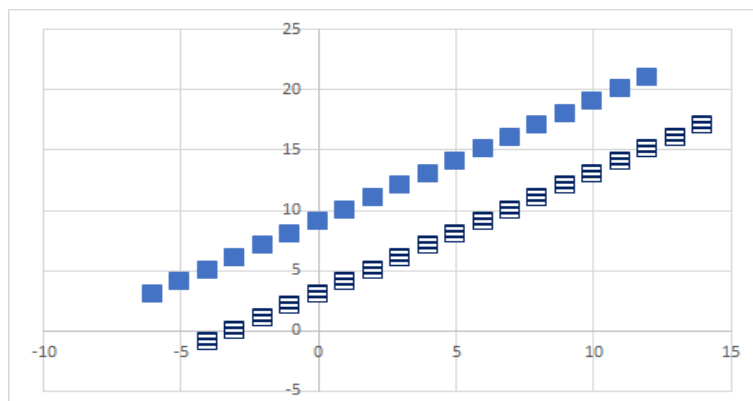Its graphic representation in a subset of $\mathbb{Z}^2$ is given by the Fig. 3:

Fig. 3. The image of the space $\mathbf{D} \subset \mathbb{Z}^2$ by the discrete translation $\mathbf{T}_{(2,-4)}$

3.1.3 Definition

The composition of two translations $\boldsymbol{T}_u$ and $\boldsymbol{T}_v$ is defined by $\boldsymbol{T}_u \ \boldsymbol{T}_v = \boldsymbol{T}_{u+v}$

The Algebraic proprieties for the set of discrete translations of isolated points with the usually composition are similar to those of translations in continuous spaces.

Let's verify the Algebraic proprieties for the set of discrete translations of isolated points with the law of composition.

a) **Stability:** In fact, let T_u and T_v two discrete translations respectively the vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ defined in $\mathbb{Z}^2$ respectively by :

$$T_u(x, y) = (x', y') \text{ where } \begin{cases} x' = x + \left\lfloor u_1 + \frac{1}{2} \right\rfloor \\ y' = y + \left\lfloor u_2 + \frac{1}{2} \right\rfloor \end{cases} \text{ and}$$

$$T_v(x, y) = (x', y') \text{ with } \begin{cases} x' = x + \left\lfloor v_1 + \frac{1}{2} \right\rfloor \\ y' = y + \left\lfloor v_2 + \frac{1}{2} \right\rfloor \end{cases}$$

The composition of two discrete translations T_u and T_v is defined by :

$$T_u \ T_v(x, y) = \left(x + \left\lfloor u_1 + v_1 + \frac{1}{2} \right\rfloor ; y + \left\lfloor u_2 + v_2 + \frac{1}{2} \right\rfloor \right)$$

is well a translation of vector $u + v$.

b) **Associativity :** for all translations T_u, T_v, T_w belonging in a set of translations with $u = (u_1, u_2)$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$ of vectors translations of $\mathbb{Z}^2$ respectively T_u, T_v, T_w. Hence:
$T_u \ (T_v \ T_w) = (T_u \ T_v) \ T_w$

c) **Existence of the neutral element:** The discrete translation T_0 of vector nul 0 is neutral element of the set of discrete translations with the law of composition.

d) **Symmetrization :** All discrete transformation T_u admits for symmetric the discrete translation T_{-u}

e) **Commutativity:** For all discrete translations T_u and T_v, we have:

$T_u \ T_v = T_v \ T_u$

3.1.4 Proposition

The set of discrete translations of isolated points with the usually composition has a structure of commutative group.

3.2 Discrete pythagorean rotation

3.2.1 Definition

A discrete Pythagorean rotation is defined by [12, 2]:

$\mathcal{RP}(k) : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = \left\lfloor \dfrac{bx - ay + \frac{b}{2}}{b + 1} \right\rfloor \\ y' = \left\lfloor \dfrac{ax + by + \frac{b}{2}}{b + 1} \right\rfloor \end{cases}$$

with a, b, k integers such as $a = 2k + 1$ and $b = 2k(k + 1)$

3.2.2 Example

Let a Pythagorean rotation defined by

$$\mathcal{RP}(1) : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{cases} x' = \left\lfloor \frac{4x - 3y + 2}{5} \right\rfloor \\ y' = \left\lfloor \frac{3x + 4y + 2}{5} \right\rfloor \end{cases} \text{ where } k = 1 \text{ an integer.}$$

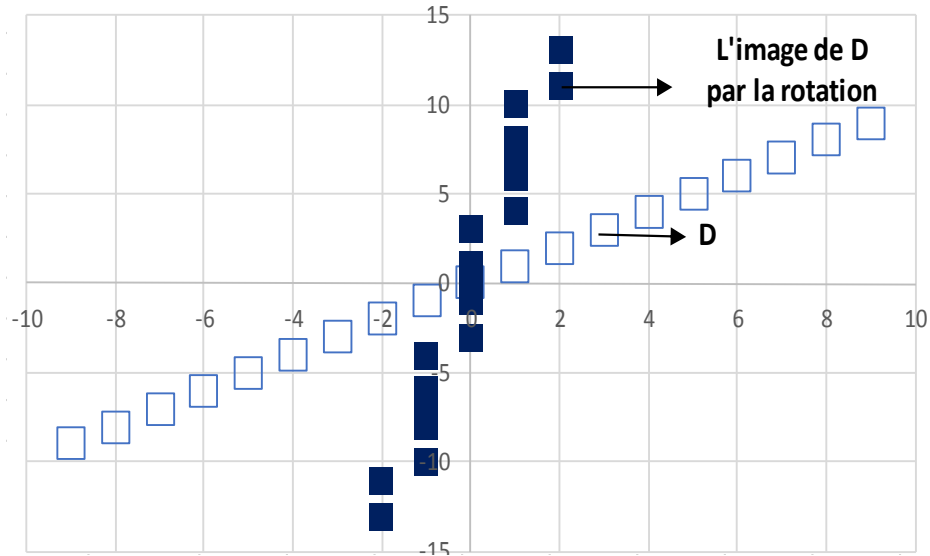Its graphic representation in a subset of $\mathbb{Z}^2$ is given by the Fig. 4:

Fig. 4. The image of the space $D \subset \mathbb{Z}^2$ by the Pythagorean where $k = 1$

3.2.3 Proposition

A Pythagorean rotation is bijective

The proof of this proposition is established in his article [13]

3.2.4 Corollary

A Pythagorean rotation is optimal for $k = -1$.

Considering the example above related to the Pythagorean rotation, we obtain for $k = -1$

$$\mathcal{RP}(-1) : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{cases} x' = -y \\ y' = -x \end{cases}$$
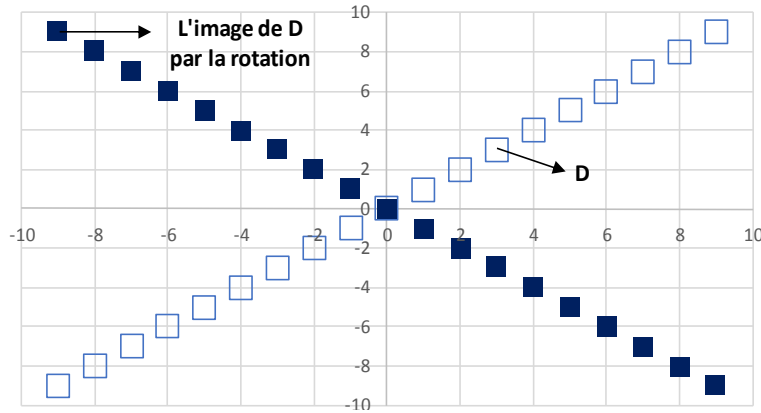
Its is show by Fig. 5:

Fig. 5. The image of the space $D \subset \mathbb{Z}^2$ by the Pythagorean rotation where $k = -1$

3.2.5 Definition

Let two Pythagorean rotations $\mathcal{RP}(k)$ and $\mathcal{RP}(k')$ where $(k, k') \in \mathbb{Z} \times \{-1\}$, the composition of the two Pythagorean rotations is defined by

$\mathcal{RP}(k) \;\; \mathcal{RP}(k') : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} \text{ with } \begin{cases} x' = \left\lfloor \dfrac{(bb'-aa')x-(ab'+ba')y+bb'+\frac{(b'-a'b)}{2}}{(b+1)(b'+1)} \right\rfloor \\[3ex] y' = \left\lfloor \dfrac{(ab'+ba')x-(bb'-aa')y+bb'+\frac{(b'-a'b)}{2}}{(b+1)(b'+1)} \right\rfloor \end{cases}$$

et a, a', b, b', k, k' the integers such as $a = 2k + 1$, $b = 2k(k+1)$, $a' = 2k' + 1$ et $b' = 2k'(k'+1)$

Let verify the Algebraic properties for the set of the Pythagorean rotations with the law of composition.

a) **Stability:** In fact, let $\mathcal{RP}(k)$ and $\mathcal{RP}(k')$ two Pythagorean rotations of the parameters k et k' respectively defined in $\mathbb{Z}^2$ by:

$\mathcal{RP}(k) : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = \left\lfloor \dfrac{bx + ay + \frac{b}{2}}{b+1} \right\rfloor \\[3ex] y' = \left\lfloor \dfrac{ax + by + \frac{b}{2}}{b+1} \right\rfloor \end{cases}$$

with a, b, k integers such as $a = 2k + 1$ and $b = 2k(k+1)$ and by

$\mathcal{RP}(k') : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} x' = \left\lfloor \dfrac{b'x+a'y+\frac{b'}{2}}{b'+1} \right\rfloor \\[3ex] y' = \left\lfloor \dfrac{a'x+b'y+\frac{b'}{2}}{b'+1} \right\rfloor \end{cases}$$

where a, b, k integers such as $a' = 2k' + 1$ and $b = 2k'(k'+1)$

The composition of two Pythagorean rotations RP(k) and RP(k') is defined by

$$\mathcal{RP}(\text{k}) \quad \mathcal{RP}(\text{k'}) : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} \text{ with } \begin{cases} x' = \left\lfloor \dfrac{(bb'-aa')x-(ab'+ba')y+bb'+\frac{(b'-a'b)}{2}}{(b+1)(b'+1)} \right\rfloor \\[4mm] y' = \left\lfloor \dfrac{(ab'+ba')x-(bb'-aa')y+bb'+\frac{(b'-a'b)}{2}}{(b+1)(b'+1)} \right\rfloor \end{cases}$$

Is a Pythagorean rotation as long as $(\text{k}, \text{k'}) \in \mathbb{Z} \times \{-1\}$

b) **Associativity.** For all Pythagorean rotations $\mathcal{RP}(\text{k}), \mathcal{RP}(\text{k'})$ and RP(k'') belonging in the set of Pythagorean rotations with $(\text{k}, \text{k'}, \text{k''}) \in \{-1\} \times \mathbb{Z} \times \{-1\}$, we have:

$$\mathcal{RP}(\text{k}) \circ \Big(\mathcal{RP}(\text{k'}) \circ \mathcal{RP}(\text{k''}) \Big) = \Big(\mathcal{RP}(\text{k}) \circ \mathcal{RP}(\text{k'}) \Big) \circ \mathcal{RP}(\text{k''})$$

c) Existence of the neutral element. The set of Pythagorean rotations never have the neutral element for the composition of transformations because there does not exist a unique value of e such that

$$\mathcal{RP}(\text{k'}) \circ \mathcal{RP}(\text{e}) = \mathcal{RP}(\text{e}) \circ \mathcal{RP}(\text{k'}) = \mathcal{RP}(\text{k})$$

d) **Symmetrization.** All Pythagorean rotations admit for symmetry the Pythagorean rotation of parameters $(-\text{k}-1)$ since $\mathcal{RP}(\text{k}) \circ \mathcal{RP}(-\text{k}-1) = \text{id}_{\mathbb{Z}}$.

e) Commutativity. For all Pythagorean rotations $\mathcal{RP}(k)$ and $\mathcal{RP}(k')$, the composition $\mathcal{RP}(k) \ \mathcal{RP}(k')$ is not commutative since the integers of factor $\dfrac{\frac{(b'-a'b)}{2}}{(b+1)(b'+1)}$ never commute in $\mathbb{Z}$.

3.2.6 Proposition

The set of Pythagorean rotations with the composition of transformations never has a structure of the commutative group.

4 Conclusion

The quasi-affine transformations do not always preserve the algebraic structures like their corresponding ones in the continuous space (affine rotation and affine translation). This article focusses on the study of Algebraic structures of the bijective QAT, in particular of discrete translations and Pythagorean rotations. The following results were found:

- A quasi-linear transformation is bijective if $\omega = gcd(|a|, |b|, |c|, |d|)$ and $\det(A) \neq 0$, and a quasi-affine transformation is bijective if $\omega = gcd(|a|, |b|, |c|, |d|, |e|, |f|)$ and $\det(A) \neq 0$.
- The set of quasi-linear bijective transformations having the usual composition group is called quasi-linear group, noted by $\lfloor GQL \rfloor$ and the set of quasi-affine bijective transformations having the usual composition group is called quasi-affine group, noted by $\lfloor GAL \rfloor$.
- The set of Pythagorean rotations with the composition of transformations never has a structure of the commutative group.
- In general, the composition of two Pythagorean rotations is not a Pythagorean rotation, unless in case where one of the parameters equal -1 and for if $k = -1$, the Pythagorean rotation is optimal

The inverses of images of a bijective quasi-affine application do not generate a tilling in the literal sense of the term, because they are made up of singletons.

Subsequent developments in this work on bijcetive QAT will focus, among other things, around:

- Study of one-to-one QAT and tilings. For a given bijective QAT we observe that the block is a singleton. Are the different block concepts still valid for one-to-one QAT?
- Study of bijective QAT in discrete spaces of higher dimension. This will lead to a generalization on one-to-one QAT.
- Study of image transformations with one-to-one QAT.

Competing Interests

Authors have declared that no competing interests exist.

References

[1]   Jacob-Da Col MA, Tellier P. Quasi-linear transformations and discrete tiilings," Theoretical computer science. 2009;1410:2126-2134.

[2]   Andres E, Jacob-Da Col MA. Discrete affine transformations", In: D. Coeurjolly, A. Montanvert and J. Chassery (Eds.), Discrete geometry and digital images, Paris: Lavoisier. 2007;173-190.

[3]   Jacob-Da Col MA. Quasi-affine applications and tilings of the discrete plane," Theoretical Computer Science. 2011;259(11-2):245-269.

[4]   Blot V, Coeujolly D. Quasi-affine Transformation in Higher dimension," 15th International Conference on Discrete Geometry for Computer Imagery, LNCS, Montreal; 2009.

[5]   Pluta K, Romon P, Kenmochi Y, Passat N. Bijective digitized rigid motions on subsets of the plane Journal of Mathematical Imaging and Vision. 2017;59:84–105.
Available:https://doi.org/10.1007/s10851-017-0706-8.

[6]   Mbuyi Kalubi C, Musesa Landa A. On Affine transformations in continuous and in discrete spaces," Congo sciences. 2020;1-8.

[7]   Kim C. Three-dimensional digital planes," IIEEE Transactions on Pattern Analysis and Machine Intelligence. 1984;6(15):639-645.

[8]   Nehlig P. Quasi-affine transformation: tilings by reciprocal images," Theorical Computer Science. 1995;156:1-38.

[9]   Jacob-Da Col MA. Applications quasi-affines", Ph.D. Thesis, University Louis Pasteur, Strasbourg; 1993.

[10]   Jacob-Da co MA, P Tellier. Quasi-linear transformations and discrete tilings," Theoretical Computer Science. 2009;410:2126-2134.

[11]   Arnaudiès J, Fraysse H. Mathematics Course 1-Algebra, Paris: Bordas; 1982.

[12]   Thibault Y. Rotations in 2D and 3D discrete spaces", France: PhD, Université Paris-Est; 2010.

[13]   Jacob-Da Col MA, Andres E. On discrete rotations", 5th Int. Workshop on Discret Geometry for Computer Science, vol. 156, Elseiver Science. 1995;1-38.