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Anti complex fuzzy subgroups under s-norms

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Abstract: In this study, we define anti complex fuzzy subgroups and normal anti complex fuzzy subgroups under s-norms and investigate some of characteristics of them. Later we introduce and study the intersection and composition of them. Next, we define the concept normality between two anti complex fuzzy subgroups by using s-norms and obtain some properties of them. Finally, we define the image and the inverse image of them under group homomorphisms.

Keywords: Group theory, fuzzy groups, norms, intersections, compositions, complex fuzzy subgroups, normal complex fuzzy subgroups.

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1. Introduction

Group theory has applications in physics, chemistry, and computer science, and even puzzles like Rubik's Cube can be represented using group theory. Fuzzy sets, proposed by Zadeh [1], are sets whose elements have degrees of membership. Rosenfeld [2] introduced fuzzy sets in the realm of group theory and formulated the concepts of fuzzy subgroups of a group. Many authors have worked on fuzzy group theory [2–4], especially, some authors considered the fuzzy subgroups with respect to norms [5–7]. Alsarahead and Ahmad [8] defined the complex fuzzy subgroup and investigate some of its characteristics. The author by using norms, investigated some properties of fuzzy algebraic structures [9–11].

In this paper, by using s-norms, we define and investigate some properties of anti complex fuzzy subgroups of group G under s-norm S as $ACFS(G)$. Also we define the composition and intersection of two $\mu_1, \mu_2 \in ACFS(G)$ and obtain some of their characteristics. Later, we introduce and investigate the normality of $\mu \in ACFS(G)$ denoted by $NACFS(G)$. Finally, we introduce the normality between two $\mu_1, \mu_2 \in ACFS(G)$ as $\mu_1 \bowtie \mu_2$ and investigate some important properties of them. By using a group homomorphism $f : G \to H$, we prove that if $\mu \in ACFS(G)$ and $\nu \in ACFS(H)$, then $f(\mu) \in ACFS(H)$ and $f^{-1}(\nu) \in ACFS(G)$. Also if $\mu \in NACFS(G)$ and $\nu \in NACFS(H)$, then we prove that $f(\mu) \in NACFS(H)$ and $f^{-1}(\nu) \in NACFS(G)$. Also we show that if $\mu_1, \mu_2 \in ACFS(G)$ such that $\mu_1 \bowtie \mu_2$, then we show that $f(\mu_1) \bowtie f(\mu_2)$ and if $\mu_1, \mu_2 \in ACFS(H)$ such that $\mu_1 \bowtie \mu_2$, then we obtain $f^{-1}(\mu_1) \bowtie f^{-1}(\mu_2)$.

2. Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. For details we refer readers to [6,12–15].

Definition 1. A group is a non-empty set G, on which there is a binary operation $(a, b) \to ab$ such that

- if a and b belong to G then ab is also in G (closure),
- $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),
- there is an element $e_G \in G$ such that $ae_G = ee_G a = a$ for all $a \in G$ (identity),
- if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e_G$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$.

Remark 1. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \to a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \to ab$ for which the identity is denoted by e.

Definition 2. Let G be an arbitrary group with a multiplicative binary operation and identity e. By a fuzzy subset of G, we mean a function from G into $[0,1]$. The set of all fuzzy subsets of G is called the $[0,1]$-power set of G and is denoted $[0,1]^G$.

Definition 3. Let X be a nonempty set. A complex fuzzy set A on X is an object having the form $A = \{(x, \mu_A(x)) | x \in X\}$, where μ_A denotes the degree of membership function that assigns each element $x \in X$, a complex number $\mu_A(x))$ lies within the unit circle in the complex plane. We shall assume that $\mu_A(x)$ will be represented by $r_A(x)e^{iw_A(x)}$, where $i = \sqrt{-1}$, and $r : X \to [0,1]$ and $w : X \to [0,2\pi]$. Note that by setting $w(x) = 0$ in the definition above, we return back to the traditional fuzzy subset. Let $\mu_1 = r_1e^{w_1}$ and $\mu_2 = r_2e^{w_2}$ be two complex numbers lie within the unit circle in the complex plane. By $\mu_1 \leq \mu_2$, we mean $r_1 \leq r_2$ and $w_1 \leq w_2$.

Definition 4. An s-norm S is a function $S : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

- $S(x,0) = x$,
- $S(x,y) \leq S(x,z)$ if $y \leq z$,
- $S(x,y) = S(y,x)$,
- $S(x,S(y,z)) = S(S(x,y),z)$

for all $x, y, z \in [0,1]$.

We say that S is idempotent if for all $x \in [0,1]$,$S(x,x) = x$.

Example 1. The basic s-norms are $S_m(x,y) = \max\{x,y\}$,$S_b(x,y) = \min\{1, x+y\}$ and $S_p(x,y) = x+y-xy$ for all $x, y \in [0,1]$. S_m is standard union, S_b is bounded sum, S_p is algebraic sum.

Lemma 1. *Let S be an s-norm. Then*

$$S(S(x,y), S(w,z)) = S(S(x,w), S(y,z)),$$

for all $x, y, w, z \in [0,1]$.

3. Anti complex fuzzy subgroups under s-norms

Definition 5. Let G be a group and $\mu : G \to [0,1]$ be a complex fuzzy set on G. Then $\mu = re^{iw}$ is said to be an anti complex fuzzy subgroup of G under s-norm S if the following conditions hold:

- $r(xy) \leq S(r(x), r(y))$,
- $r(x^{-1}) \leq r(x)$,
- $w(xy) \leq \max\{w(x), w(y)\}$,
- $w(x^{-1}) \leq w(x)$,

for all $x, y \in G$. The set of all anti complex fuzzy subgroups of G under s-norm S is denoted by $ACFS(G)$.

Example 2. Let $G = \{0, a, b, c\}$ be the Klein's group. Every element is its own inverse, and the product of any two distinct non-identity elements is the remaining non-identity element. Thus the Klein 4-group admits the elegant presentation $a^2 = b^2 = c^2 = abc = 0$. Define $r : G \to [0,1]$ by

$$r(x) = \begin{cases} 0.35 & \text{if } x = a, \\ 0.45 & \text{if } x = b, \\ 0.65 & \text{if } x = c, \\ 0.85 & \text{if } x = 0, \end{cases}$$

and $w : G \to [0, 2\pi]$ by

$$w(x) = \begin{cases} 0.4\pi & \text{if } x = a, \\ 0.4\pi & \text{if } x = b, \\ 0.5\pi & \text{if } x = c, \\ 0.6\pi & \text{if } x = 0. \end{cases}$$

Let $S(a, b) = S_p(a, b) = a + b - ab$ for all $a, b \in [0, 1]$, then $\mu(x) = r(x)e^{iw(x)} \in ACFS(G)$ for all $x \in G$.

Proposition 1. *Let $\mu = re^{iw} \in ACFS(G)$ such that C be idempotent s-norm. Then*

- *$\mu(e) \leq \mu(x)$ for all $x \in G$,*
- *$\mu(x^n) \leq \mu(x)$ for all $x \in G$ and $n \geq 1$,*
- *$\mu(x) = \mu(x^{-1})$ for all $x \in G$.*

Proof. Let $\mu = re^{iw} \in ACFS(G)$ and $x \in G$ and $n \geq 1$. Then

$$\begin{aligned}
(1)r(e) &= r(xx^{-1}) \leq S(r(x), r(x^{-1})) \leq S(r(x), r(x)) = r(x), \\
w(e) &= w(xx^{-1}) \leq \max\{w(x), w(x^{-1})\} \leq \max\{w(x), w(x)\} = w(x), \\
\mu(e) &= r(e)e^{iw(x)} \leq r(x)e^{iw(x)} = \mu(x), \\
r(x^n) &= r(\underbrace{xx...x}_{n}) \leq S(\underbrace{r(x), r(x), ..., r(x)}_{n}) = r(x), \\
w(x^n) &= w(\underbrace{xx...x}_{n}) \leq \max\{\underbrace{r(x), r(x), ..., r(x)}_{n}\} = w(x), \\
\mu(x^n) &= r(x^n)e^{iw(x^n)} \leq r(x)e^{iw(x)} = \mu(x), \\
r(x) &= r((x^{-1}))^{-1} \leq r(x^{-1}) \leq r(x), \\
r(x) &= r(x^{-1}), \\
w(x) &= w((x^{-1}))^{-1} \leq w(x^{-1}) \leq w(x), \\
w(x) &= w(x^{-1}), \\
\mu(x) &= r(x)e^{iw(x)} = r(x^{-1})e^{iw(x^{-1})} = \mu(x^{-1}).
\end{aligned}$$

□

Proposition 2. *Let $\mu = re^{iw} \in ACFS(G)$ and $x \in G$ such that S be idempotent s-norm. Then $\mu(xy) = \mu(y)$ $\forall y \in G$ if and only if $\mu(x) = \mu(e)$.*

Proof. Let $\mu(xy) = \mu(y)$ $\forall y \in G$. As $y = e$, so $\mu(x) = \mu(e)$. Conversely, let $\mu(x) = \mu(e)$, then $r(x) = r(e)$ and $w(x) = w(e)$. From Proposition 1, we get $r(x) \leq r(y)$ and $r(x) \leq r(xy)$. Also $w(x) \leq w(y)$ and $w(x) \leq w(xy)$. Now $r(xy) \leq S(r(x), r(y)) \leq S(r(y), r(y)) = r(y) = r(x^{-1}xy) \leq S(r(x), r(xy)) \leq S(r(xy), r(xy)) = r(xy)$.

Also $w(xy) \leq \max\{w(x), w(y)\} \leq \max\{w(y), w(y)\} = w(y) = w(x^{-1}xy) \leq \max\{w(x), w(xy)\} \leq \max\{w(xy), w(xy)\} = w(xy)$. Therefore $\mu(xy) = r(xy)e^{iw(xy)} = r(y)e^{iw(y)} = \mu(y)$. □

Definition 6. Let G be a set and $\mu_1 = r_1e^{iw_1}$, $\mu_2 = r_2e^{iw_2}$ be two complex fuzzy sets on G. Denote the composition of μ_1 and μ_2 as $\mu_1 \circ \mu_2 = (r_1 \circ r_2)e^{i(w_1 \circ w_2)}$ such that $r_1 \circ r_2 : G \to [0, 1]$ and $w_1 \circ w_2 : G \to [0, 2\pi]$ and define by $(\mu_1 \circ \mu_2)(x) = (r_1 \circ r_2)(x)e^{i(w_1 \circ w_2)(x)}(x)$ such that

$$(r_1 \circ r_2)(x) = \begin{cases} \inf\limits_{x=ab} S(r_1(a), r_2(b)) & \text{if } x = ab, \\ 0 & \text{if } x \neq ab, \end{cases}$$

and

$$(w_1 \circ w_2)(x) = \begin{cases} \max\limits_{x=ab}\{w_1(a), w_2(b)\} & \text{if } x = ab, \\ 0 & \text{if } x \neq ab, \end{cases}$$

thus $(\mu_1 \circ \mu_2)(x) = \inf_{x=ab} S(r_1(a), r_2(b))e \cdot i \max_{x=ab}\{w_1(a), w_2(b)\}$.

Proposition 3. *Let μ^{-1} be the inverse of μ such that $\mu^{-1}(x) = \mu(x^{-1})$. Then $\mu \in ACFS(G)$ if and only if μ satisfies the following conditions:*

(1) $\mu \leq \mu \circ \mu$;
(2) $\mu^{-1} = \mu$.

Proof. Let $x, y, z \in G$ with $x = yz$ and $\mu \in ACFS(G)$. Then

$$\begin{aligned} r(x) &= r(yz) \leq S(r(y), r(z)) = (r \circ r)(x), \\ w(x) &= w(yz) \leq \max\{r(y), r(z)\} = (w \circ w)(x), \\ \mu(x) &= r(x)e^{iw(x)} \leq (r \circ r)(x)e^{i(w \circ w)(x)} = (\mu \circ \mu)(x), \end{aligned}$$

so $\mu \leq \mu \circ \mu$.

Also from Proposition 1, for all $x \in G$, we have that $\mu^{-1}(x) = \mu(x^{-1}) = \mu(x)$ and so $\mu^{-1} = \mu$. Conversely let $\mu \leq \mu \circ \mu$ and $\mu^{-1} = \mu$. We prove that $\mu \in ACFS(G)$. As $\mu \leq \mu \circ \mu$ so $r(x) \leq (r \circ r)(x)$ and $w(x) \leq (w \circ w)(x)$. Thus

$$r(yz) = r(x) \leq (r \circ r)(x) = \inf_{x=yz} S(r(y), r(z)) \leq S(r(y), r(z))$$

and

$$w(yz) = w(x) \geq (w \circ w)(x) = \max_{x=yz}\{w(y), w(z)\} \geq \{w(y), w(z)\}.$$

Since $\mu^{-1} = \mu$ so $r^{-1}(x) = r(x)$ and $w^{-1}(x) = w(x)$. Therefore $r(x^{-1}) = r^{-1}(x) = r(x)$ and $w(x^{-1}) = w^{-1}(x) = w(x)$. Then $\mu \in ACFS(G)$. □

Corollary 1. *Let $\mu_1, \mu_2 \in ACFS(G)$ and G be commutative group. Then $\mu_1 \circ \mu_2 \in ACFS(G)$ if and only if $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$.*

Proof. As $\mu_1, \mu_2 \in CFST(G)$ and $\mu_1 \circ \mu_2 \in ACFS(G)$, so from Proposition 3, we get $\mu_1^{-1} = \mu_1$ and $\mu_2^{-1} = \mu_2$ and $(\mu_2 \circ \mu_1)^{-1} = \mu_2 \circ \mu_1$. Then $\mu_1 \circ \mu_2 = \mu_1^{-1} \circ \mu_2^{-1} = (\mu_2 \circ \mu_1)^{-1} = \mu_2 \circ \mu_1$. Conversely, let $\mu_1 \circ \mu_2 = \mu_2 \circ \mu_1$, then $(\mu_1 \circ \mu_2) \circ (\mu_1 \circ \mu_2) = \mu_1 \circ (\mu_2 \circ \mu_1) \circ \mu_2 = \mu_1 \circ (\mu_1 \circ \mu_2) \circ \mu_2 = (\mu_1 \circ \mu_1) \circ (\mu_2 \circ \mu_2) \geq \mu_1 \circ \mu_2$. Also $(\mu_1 \circ \mu_2)^{-1} = (\mu_2 \circ \mu_1)^{-1} = \mu_1^{-1} \circ \mu_2^{-1} = \mu_1 \circ \mu_2$. Then the Proposition 3 gives us that $\mu_1 \circ \mu_2 \in ACFS(G)$. □

Definition 7. Let $\mu_1 = r_1 e^{iw_1} \in ACFS(G)$ and $\mu_2 = r_2 e^{iw_2} \in ACFS(G)$. Define the intersection $\mu_1 \cap \mu_2$ as $\mu_1 \cap \mu_2 = r_1 e^{iw_1} \cap r_2 e^{iw_2} = (r_1 \cap r_2) e^{i(w_1 \cap w_2)}$ such that $r_1 \cap r_2 : G \to [0, 1]$ and $w_1 \cap w_2 : G \to [0, 2\pi]$ and for all $x \in G$, define $(r_1 \cap r_2)(x) = S(r_1(x), r_2(x))$ and $(w_1 \cap r_2)(x) = \max\{w_1(x), w_2(x)\}$.

Proposition 4. *Let $\mu_1 = r_1 e^{iw_1} \in ACFS(G)$ and $\mu_2 = r_2 e^{iw_2} \in ACFS(G)$. Then $\mu_1 \cap \mu_2 \in ACFS(G)$.*

Proof. (1) Let $g_1, g_2 \in G$. Then

$$\begin{aligned} (r_1 \cap r_2)(g_1 g_2) &= S(r_1(g_1 g_2), r_2(g_1 g_2)) \leq S(S(r_1(g_1), r_1(g_2)), S(r_2(g_1), r_2(g_2))) \\ &= S(S(r_1(g_1), r_2(g_1)), T(r_1(g_2), r_2(g_2))) \\ &= S((r_1 \cap r_2)(g_1), (r_1 \cap r_2)(g_2)), \end{aligned}$$

and thus $(r_1 \cap r_2)(g_1 g_2) \leq S((r_1 \cap r_2)(g_1), (r_1 \cap r_2)(g_2))$.

(2) If $g \in G$, then

$$\begin{aligned} (r_1 \cap r_2)(g^{-1}) = S(r_1(g^{-1}), r_2(g^{-1})) &\leq S(r_1(g), r_2(g)) = (r_1 \cap r_2)(g) \\ (r_1 \cap r_2)(g^{-1}) &\geq (r_1 \cap r_2)(g). \end{aligned}$$

(3) Let $g_1, g_2 \in G$. Then

$$\begin{aligned}(w_1 \cap w_2)(g_1g_2) &= \max\{w_1(g_1g_2), w_2(g_1g_2)\}\\ &\leq \max\{\max\{w_1(g_1), w_1(g_2)\}, \max\{w_2(g_1), w_2(g_2)\}\}\\ &= \max\{\max\{w_1(g_1), w_2(g_1)\}, \max\{w_1(g_2), w_2(g_2))\}\\ &= \max\{(w_1 \cap w_2)(g_1), (w_1 \cap w_2)(g_2))\\ (w_1 \cap w_2)(g_1g_2) &\leq \max\{(w_1 \cap w_2)(g_1), (w_1 \cap w_2)(g_2)).\end{aligned}$$

(4) Let $g \in G$ so

$$\begin{aligned}(w_1 \cap w_2)(g^{-1}) &= \max\{w_1(g^{-1}), w_{21}(g^{-1})\}\\ &\leq \max\{w_1(g), w_2(g)\}\\ &= (w_1 \cap w_2)(g)\\ (w_1 \cap w_2)(g^{-1}) &\leq (w_1 \cap w_2)(g).\end{aligned}$$

Thus from (1)-(4) we give that $\mu_1 \cap \mu_2 \in ACFS(G)$. □

Corollary 2. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq ACFS(G)$ then $\mu = \cap_{i \in I_n}\mu_i \in ACFS(G)$.*

Definition 8. $\mu \in ACFS(G)$ is called normal if for all $x, y \in G$ we have that $\mu(xyx^{-1}) = \mu(y)$. The set of all normal anti complex fuzzy subgroups of G under s-norm S ia denoted by $NACFS(G)$.

Proposition 5. *Let $\mu_1 = r_1e^{iw_1} \in NACFS(G)$ and $\mu_2 = r_2e^{iw_2} \in NACFS(G)$. Then $\mu_1 \cap \mu_2 \in NACFS(G)$.*

Proof. From Proposition 4 we will have that $\mu_1 \cap \mu_2 \in ACFS(G)$. Let $g_1, g_2 \in G$ then

$$\begin{aligned}(r_1 \cap r_2)(g_1g_2g_1^{-1}) &= S(r_1(g_1g_2g_1^{-1}), r_2(g_1g_2g_1^{-1})) = S(r_1(g_2), r_2(g_2)) = (r_1 \cap r_2)(g_2)\\ (w_1 \cap w_2)(g_1g_2g_1^{-1}) &= \max\{w_1(g_1g_2g_1^{-1}), w_2(g_1g_2g_1^{-1})\} = \max\{w_1(g_2), w_2(g_2)\} = (w_1 \cap w_2)(g_2)\\ (\mu_1 \cap \mu_2)(g_1g_2g_1^{-1}) &= (r_1 \cap r_2)(g_1g_2g_1^{-1})e^{i(w_1 \cap w_2)(g_1g_2g_1^{-1})} = (r_1 \cap r_2)(g_2)e^{i(w_1 \cap w_2)(g_2)} = (\mu_1 \cap \mu_2)(g_2)\end{aligned}$$

and therefore $\mu_1 \cap \mu_2 \in NACFS(G)$. □

Corollary 3. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq NACFS(G)$, then $\mu = \cap_{i \in I_n}\mu_i \in NACFS(G)$.*

Definition 9. Let $\mu_1 = r_1e^{iw_1} \in ACFS(G)$ and $\mu_2 = r_2e^{iw_2} \in ACFS(G)$ such that $\mu_1 \subseteq \mu_2$. We say that μ_1 is normal of the μ_2, written $\mu_1 \bowtie \mu_2$, if $r_1(g_1g_2g_1^{-1}) \leq S(r_1(g_2), r_2(g_1))$ *and* $w_1(g_1g_2g_1^{-1}) \leq \max\{w_1(g_2), w_2(g_1)\}$ for all $g_1, g_2 \in G$.

Proposition 6. *If S be idempotent s-norm, then every $\mu = re^{iw} \in ACFS(G)$ will be normal of itself.*

Proof. Let $g_1, g_2 \in G$ and $\mu = re^{iw} \in ACFS(G)$, then

$$\begin{aligned}r(g_1g_2g_1^{-1}) &\leq S(r(g_1), r(g_2g_1^{-1}))\\ &\leq S(r(g_1), S(r(g_2), r(g_1^{-1})))\\ &\leq S(r(g_1), S(r(g_2), r(g_1)))\\ &= S(r(g_2), S(r(g_1), r(g_1)))\\ &= S(r(g_2), r(g_1))\\ (g_1g_2g_1^{-1}) &\leq S(r(g_2), r(g_1)).\end{aligned}$$

Also

$$\begin{aligned} w(g_1g_2g_1^{-1}) &\leq \max\{w(g_1), w(g_2g_1^{-1}))\} \\ &\leq \max\{w(g_1), \max\{w(g_2), w(g_1^{-1})\}\} \\ &\leq \max\{w(g_1), \max\{w(g_2), w(g_1)))\\ &= \max\{w(g_2), \max\{w(g_1), w(g_1)\}\} \\ &= \max\{w(g_2), w(g_1)\} \\ w(g_1g_2g_1^{-1}) &\leq \max\{w(g_2), w(g_1)\}. \end{aligned}$$

Therefore $\mu = re^{iw} \bowtie \mu = re^{iw}$. □

Proposition 7. *Let $\mu_1 = r_1e^{iw_1} \in NACFS(G)$ and $\mu_2 = r_2e^{iw_2} \in ACFS(G)$ such that S be idempotent s-norm, then $\mu_1 \cap \mu_2 \bowtie \mu_2$.*

Proof. As Proposition 4 $(\mu_1 \cap \mu_2) \leq \mu_2$ and $(\mu_1 \cap \mu_2) \in ACFS(G)$. Let $g_1, g_2 \in G$ and $\mu_1 \cap \mu_2 = (r_1 \cap r_2)e^{i(w_1 \cap w_2)}$, then

$$\begin{aligned} (r_1 \cap r_2)(g_1g_2g_1^{-1}) &= S(r_1(g_1g_2g_1^{-1}), r_2(g_1g_2g_1^{-1})) \\ &= S(r_1(g_2), r_2(g_1g_2g_1^{-1})) \\ &\leq S(r_1(g_2), S(r_2(g_1g_2), r_2(g_1^{-1}))) \\ &\leq S(r_1(g_2), S(r_2(g_1g_2), r_2(g_1))) \\ &\leq S(r_1(g_2), S(S(r_2(g_1), r_2(g_2)), r_2(g_1))) \\ &= S(r_1(g_2), S(S(r_2(g_1), r_2(g_1)), r_2(g_2))) \\ &= S(r_1(g_2), S(r_2(g_1), r_2(g_2))) \\ &= S(S(r_1(g_2), r_2(g_2)), r_2(g_1)) \\ &= S((r_1 \cap r_2)(g_2), r_2(g_1)), \end{aligned}$$

and thus $(r_1 \cap r_2)(g_1g_2g_1^{-1}) \leq S((r_1 \cap r_2)(g_2), r_2(g_1))$.

Also

$$\begin{aligned} (w_1 \cap w_2)(g_1g_2g_1^{-1}) &= \max\{w_1(g_1g_2g_1^{-1}), w_2(g_1g_2g_1^{-1})\} \\ &= \max\{w_1(g_2), w_2(g_1g_2g_1^{-1})\} \\ &\leq \max\{w_1(g_2), \max\{w_2(g_1g_2), w_2(g_1^{-1})\}\} \\ &\leq \max\{w_1(g_2), \max\{w_2(g_1g_2), w_2(g_1)\}\} \\ &\leq \max\{w_1(g_2), \max\{\max\{w_2(g_1), w_2(g_2)\}, w_2(g_1)\}\} \\ &= \max\{w_1(g_2), \max\{\max\{w_2(g_1), w_2(g_1))\}, w_2(g_2)\}\} \\ &= \max\{w_1(g_2), \max\{w_2(g_1), w_2(g_2)\}\} \\ &= \max\{\max\{w_1(g_2), w_2(g_2)\}, w_2(g_1)\} \\ &= \max\{(w_1 \cap w_2)(g_2), w_2(g_1)\}, \end{aligned}$$

and then $(w_1 \cap w_2)(g_1g_2g_1^{-1}) \leq \max\{(w_1 \cap w_2)(g_2), w_2(g_1)\}$. Therefore $\mu_1 \cap \mu_2 = (r_1 \cap r_2)e^{i(w_1 \cap w_2)} \bowtie \mu_2$. □

Proposition 8. *Let $\mu_1 = r_1e^{iw_1} \in ACFS(G)$ and $\mu_2 = r_2e^{iw_2} \in ACFS(G)$ and $\mu_3 = r_3e^{iw_3} \in ACFS(G)$ and S be idempotent s-norm. If $\mu_1 \bowtie \mu_3$ and $\mu_2 \bowtie \mu_3$, then $\mu_1 \cap \mu_2 \bowtie \mu_3$.*

Proof. By Proposition 4, we have $\mu_1 \cap \mu_2 \in ACFS(G)$ and $\mu_1 \cap \mu_2 \leq \mu_3$. Let $g_1, g_2 \in G$. As $\mu_1 \bowtie \mu_3$, so $r_1(g_1g_2g_1^{-1}) \leq S(r_1(g_2), r_3(g_1))$ and $w_1(g_1g_2g_1^{-1}) \leq \max\{r_1(g_2), r_3(g_1)\}$ and as $\mu_2 \bowtie \mu_3$ so $r_2(g_1g_2g_1^{-1}) \leq S(r_2(g_2), r_3(g_1))$ and $w_2(g_1g_2g_1^{-1}) \leq \max\{w_2(g_2), w_3(g_1)\}$. Now

$$\begin{aligned}(r_1 \cap r_2)(g_1 g_2 g_1^{-1}) &= S(r_1(g_1 g_2 g_1^{-1}), r_2(g_1 g_2 g_1^{-1})) \\ &\leq S(S(r_1(g_2), r_3(g_1)), S(r_2(g_2), r_3(g_1))) \\ &= S(S(r_1(g_2), r_2(g_2)), S(r_3(g_1), r_3(g_1))) \\ &= S(S(r_1(g_2), r_2(g_2)), r_3(g_1)) \\ &= S((r_1 \cap r_2)(g_2), r_3(g_1)),\end{aligned}$$

and then $(r_1 \cap r_2)(g_1 g_2 g_1^{-1}) \leq S((r_1 \cap r_2)(g_2), r_3(g_1))$. Also

$$\begin{aligned}(w_1 \cap w_2)(g_1 g_2 g_1^{-1}) &= \max\{w_1(g_1 g_2 g_1^{-1}), w_2(g_1 g_2 g_1^{-1})\} \\ &\geq \max\{\max\{w_1(g_2), w_3(g_1)\}, \max\{w_2(g_2), w_3(g_1)\}\} \\ &= \max\{\max\{w_1(g_2), w_2(g_2)\}, \max\{w_3(g_1), w_3(g_1)\}\} \\ &= \max\{\max\{w_1(g_2), w_2(g_2)\}, w_3(g_1)\} \\ &= \max\{(w_1 \cap w_2)(g_2), w_3(g_1)\},\end{aligned}$$

and so $(w_1 \cap w_2)(g_1 g_2 g_1^{-1}) \leq \max\{(w_1 \cap w_2)(g_2), w_3(g_1)\}$. Thus $\mu_1 \cap \mu_2 = (r_1 \cap r_2)e^{i(w_1 \cap w_2)} \bowtie \mu_3$. □

Corollary 4. *Let $I_n = \{1, 2, ..., n\}$ and $\{\mu_i \mid i \in I_n\} \subseteq ACFS(G)$ such that $\{\mu_i \mid i \in I_n\} \bowtie \xi$. Then $\mu = \cap_{i \in I_n} \mu_i \bowtie \xi$.*

4. Group homomorphisms and anti complex fuzzy subgroups under s-norms

Definition 10. Let $f : G \to H$ be a mapping such that $\mu_G = r_G e^{iw_G}$ and $\mu_H = r_H e^{iw_H}$ be two complex fuzzy sets on G and H, respectively. Define $f(\mu_G) : H \to [0,1]$ as $f(\mu_G) = f(r_G e^{iw_G}) = f(r_G) e^{if(w_G)}$ such that for all $h \in H$ we define $f(r_G)(h) = \inf\{r_G(g) \mid g \in G, f(g) = h\}$ and $f(w_G)(h) = \inf\{w_G(g) \mid g \in G, f(g) = h\}$. Also define $f^{-1}(\mu_H) : G \to [0,1]$ as $f^{-1}(r_H e^{iw_H}) = f^{-1}(r_H) e^{if^{-1}(w_H)}$ such that for all $g \in G$, we define $f^{-1}(r_H e^{iw_H})(g) = r_H(f(g)) e^{iw_H(f(g))}$.

Proposition 9. *Let $\mu_G = r_G e^{iw_G} \in ACFS(G)$ and $f : G \to H$ be a group homomorphism, then $f(\mu_G) \in ACFS(H)$.*

Proof. (1) Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ such that $h_1 = f(g_1)$ and $h_2 = f(g_2)$. Then

$$\begin{aligned}f(r_G)(h_1 h_2) &= \inf\{r_G(g_1 g_2) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &\leq \inf\{S(r_G(g_1), r_G(g_2)) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &= S(\inf\{r_G(g_1) \mid g_1 \in G, f(g_1) = h_1\}, \inf\{r_G(g_2) \mid g_2 \in G, f(g_2) = h_2\}) \\ &= S(f(r_G)(h_1), f(r_G)(h_2)),\end{aligned}$$

and so $f(r_G)(h_1 h_2) \leq S(f(r_G)(h_1), f(r_G)(h_2))$.

(2) Let $h \in H$ and $g \in G$ such that $h = f(g)$. Then

$$\begin{aligned}f(r_G)(h^{-1}) &= \inf\{r_G(g^{-1}) \mid g^{-1} \in G, f(g^{-1}) = h^{-1}\} \\ &\leq \inf\{r_G(g) \mid g \in G, f^{-1}(g) = h^{-1}\} \\ &= \inf\{r_G(g) \mid g \in G, f(g) = h\} = f(r_G)(h),\end{aligned}$$

and so $f(r_G)(h^{-1}) \leq f(r_G)(h)$.

(3) Let $h_1, h_2 \in H$ and $g_1, g_2 \in G$ such that $h_1 = f(g_1)$ and $h_2 = f(g_2)$. Then

$$\begin{aligned}f(w_G)(h_1 h_2) &= \inf\{w_G(g_1 g_2) \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &\leq \inf\{\max\{w_G(g_1), w_G(g_2)\} \mid g_1, g_2 \in G, f(g_1) = h_1, f(g_2) = h_2\} \\ &= \max\{\inf\{w_G(g_1) \mid g_1 \in G, f(g_1) = h_1\}, \inf\{w_G(g_2) \mid g_2 \in G, f(g_2) = h_2\}\} \\ &= \max\{f(w_G)(h_1), f(w_G)(h_2)\},\end{aligned}$$

and thus $f(w_G)(h_1h_2) \geq \max\{f(w_G)(h_1), f(w_G)(h_2)\}$.

(4) Let $h \in H$ and $g \in G$ such that $h = f(g)$. Now

$$\begin{aligned} f(w_G)(h^{-1}) &= \inf\{w_G(g^{-1}) \mid g^{-1} \in G, f(g^{-1}) = h^{-1}\} \\ &\geq \inf\{w_G(g) \mid g^{-1} \in G, f^{-1}(g) = h^{-1}\} \\ &= \inf\{w_G(g) \mid g \in G, f(g) = h\} = f(w_G)(h), \end{aligned}$$

and therefore $f(w_G)(h^{-1}) \geq f(w_G)(h)$.

Thus (1) - (4) mean that $f(\mu_G) = f(r_G e^{iw_G}) = f(r_G)e^{if(w_G)} \in ACFS(H)$. □

Proposition 10. *Let $\mu_H = r_H e^{iw_H} \in ACFS(H)$ and $f : G \to H$ be a group homomorphism, then $f^{-1}(\mu_H) \in ACFS(G)$.*

Proof. (1) Let $g_1, g_2 \in G$, then $f^{-1}(r_H)(g_1g_2) = r_H(f(g_1g_2)) = r_H(f(g_1)f(g_2)) \leq S(r_H(f(g_1)), r_H(f(g_2))) = S(f^{-1}(r_H)(g_1), f^{-1}(r_H)(g_2))$, and then $f^{-1}(r_H)(g_1g_2) \leq S(f^{-1}(r_H)(g_1), f^{-1}(r_H)(g_2))$.

(2) Let $g \in G$, then $f^{-1}(r_H)(g^{-1}) = r_H(f(g^{-1})) = r_H(f^{-1}(g)) \leq r_H(f(g)) = f^{-1}(r_H)(g)$, and thus $f^{-1}(r_H)(g^{-1}) \leq f^{-1}(r_H)(g)$.

(3) Let $g_1, g_2 \in G$, so $f^{-1}(w_H)(g_1g_2) = w_H(f(g_1g_2)) = w_H(f(g_1)f(g_2)) \leq \max\{w_H(f(g_1)), w_H(f(g_2))\} = \max\{f^{-1}(w_H)(g_1), f^{-1}(w_H)(g_2)\}$, and then $f^{-1}(w_H)(g_1g_2) \leq \max\{f^{-1}(w_H)(g_1), f^{-1}(w_H)(g_2))\}$.

(4) Let $g \in G$, then $f^{-1}(w_H)(g^{-1}) = w_H(f^{-1}(g)) \leq w_H(f(g)) = f^{-1}(w_H)(g)$ and then $f^{-1}(w_H)(g^{-1}) \leq f^{-1}(w_H)(g)$.

Therefore (1)-(4) give us $f^{-1}(r_H e^{iw_H})(g) = r_H(f(g))e^{iw_H(f(g))} \in ACFS(G)$. □

Proposition 11. *Let $\mu_G = r_G e^{iw_G} \in NACFS(G)$ and $f : G \to H$ be a group homomorphism. Then $f(\mu_G) \in NACFS(H)$.*

Proof. From Proposition 9, we have $f(\mu_G) \in ACFS(H)$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$ such that $f(g_1) = h_1$ and $f(g_2) = h_2$. Now

$$\begin{aligned} f(r_G)(h_1h_2h_1^{-1}) &= \inf\{r_G(g_1g_2g_1^{-1}) \mid f(g_1g_2g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \inf\{r_G(g_2) \mid f(g_1)f(g_2)f(g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \inf\{r_G(g_2) \mid f(g_1)f(g_2)f^{-1}(g_1) = h_1h_2h_1^{-1}\} \\ &= \inf\{r_G(g_2) \mid f(g_2) = h_2\} = f(r_G)(h_2). \end{aligned}$$

Also

$$\begin{aligned} f(w_G)(h_1h_2h_1^{-1}) &= \inf\{w_G(g_1g_2g_1^{-1}) \mid f(g_1g_2g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \inf\{w_G(g_2) \mid f(g_1)f(g_2)f(g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \inf\{w_G(g_2) \mid f(g_1)f(g_2)f^{-1}(g_1) = h_1h_2h_1^{-1}\} \\ &= \inf\{w_G(g_2) \mid f(g_2) = h_2\} = f(w_G)(h_2). \end{aligned}$$

Then $f(\mu_G)(h_1h_2h_1^{-1}) = f(r_G)(h_1h_2h_1^{-1})e^{if(w_G)(h_1h_2h_1^{-1})} = f(r_G)(h_2)e^{if(w_G)(h_2)} = f(\mu_G)(h_2)$. and so $f(\mu_G) \in NACFS(H)$. □

Proposition 12. *Let $\mu_H = r_H e^{iw_H} \in NACFS(H)$ and $f : G \to H$ be a group homomorphism, then $f^{-1}(\mu_H) \in NACFS(G)$.*

Proof. Using Proposition 10, we get $f^{-1}(\mu_H) \in ACFS(G)$. Let $g_1, g_2 \in G$, then $f^{-1}(r_H)(g_1g_2g_1^{-1}) = r_H(f(g_1g_2g_1^{-1})) = r_H(f(g_1)f(g_2)f(g_1^{-1})) = r_H(f(g_1)f(g_2)f^{-1}(g_1)) = r_H(f(g_2)) = f^{-1}(r_H)(g_2)$. Also $f^{-1}(w_H)(g_1g_2g_1^{-1}) = w_H(f(g_1g_2g_1^{-1})) = w_H(f(g_1)f(g_2)f(g_1^{-1})) = w_H(f(g_1)f(g_2)f^{-1}(g_1)) = w_H(f(g_2)) = f^{-1}(w_H)(g_2)$. Thus $f^{-1}(\mu_H)(g_1g_2g_1^{-1}) = f^{-1}(r_H)(g_1g_2g_1^{-1})e^{if^{-1}(w_H)(g_1g_2g_1^{-1})} = f^{-1}(r_H)(g_2)e^{if^{-1}(w_H)(g_2)} = f^{-1}(\mu_H)(g_2)$ and thus $f^{-1}(\mu_H) \in NACFS(G)$. □

Proposition 13. *Let* $\mu_1 = r_1 e^{iw_1} \in ACFS(G)$ *and* $\mu_2 = r_2 e^{iw_2} \in ACFS(G)$ *and* $f : G \to H$ *be a group homomorphism. If* $\mu_1 \bowtie \mu_2$, *then* $f(\mu_1) \bowtie f(\mu_2)$.

Proof. We know that $f(\mu_1) = f(r_1)e^{if(w_1)}$ and $f(\mu_2) = f(r_2)e^{if(w_2)}$. By Proposition 9, we have $f(\mu_1) \in ACFS(H)$ and $f(\mu_2) \in ACFS(H)$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$ such that $f(g_1) = h_1$ and $f(g_2) = h_2$. Since $\mu_1 \bowtie \mu_2$ so $r_1(g_1g_2g_1^{-1}) \leq S(r_1(g_2), r_2(g_1))$ and $w_1(g_1g_2g_1^{-1}) \leq \max\{w_1(g_2), w_2(g_1)\}$. Now

$$\begin{aligned} f(r_1)(h_1h_2h_1^{-1}) &= \inf\{r_1(g_1g_2g_1^{-1}) \mid f(g_1g_2g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &\leq \inf\{S(r_1(g_2), r_2(g_1)) \mid f(g_1)f(g_2)f(g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \inf\{T(r_1(g_2), r_2(g_1)) \mid f(g_1)f(g_2)f^{-1}(g_1) = h_1h_2h_1^{-1}\} \\ &= S(\inf\{r_1(g_2) \mid f(g_2) = h_2\}, \inf\{r_2(g_1) \mid f(g_1) = h_1\}) \\ &= S(f(r_1)(h_2), f(r_2)(h_1)), \end{aligned}$$

and then $f(r_1)(h_1h_2h_1^{-1}) \leq S(f(r_1)(h_2), f(r_2)(h_1))$. Also

$$\begin{aligned} f(w_1)(h_1h_2h_1^{-1}) &= \inf\{w_1(g_1g_2g_1^{-1}) \mid f(g_1g_2g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &\leq \inf\{\max\{w_1(g_2), w_2(g_1)\} \mid f(g_1)f(g_2)f(g_1^{-1}) = h_1h_2h_1^{-1}\} \\ &= \inf\{\max\{w_1(g_2), w_2(g_1)\} \mid f(g_1)f(g_2)f^{-1}(g_1) = h_1h_2h_1^{-1}\} \\ &= \max\{\inf\{w_1(g_2) \mid f(g_2) = h_2\}, \inf\{w_2(g_1) \mid f(g_1) = h_1\}\} \\ &= \max\{f(w_1)(h_2), f(w_2)(h_1)\}, \end{aligned}$$

and so $f(w_1)(h_1h_2h_1^{-1}) \leq \max\{f(w_1)(h_2), f(w_2)(h_1)\}$. Hence $f(\mu_1) \bowtie f(\mu_2)$. □

Proposition 14. *Let* $\mu_1 = r_1 e^{iw_1} \in ACFS(H)$, $\mu_2 = r_2 e^{iw_2} \in ACFS(H)$ *and* $f : G \to H$ *be a group homomorphism. If* $\mu_1 \bowtie \mu_2$, *then* $f^{-1}(\mu_1) \bowtie f^{-1}(\mu_2)$.

Proof. Let $f^{-1}(\mu_1) = f^{-1}(r_1)e^{if^{-1}(w_1)}$ and $f^{-1}(\mu_2) = f^{-1}(r_2)e^{if^{-1}(w_2)}$. From Proposition 10, we obtain $f^{-1}(\mu_1) \in ACFS(G)$ and $f^{-1}(\mu_2) \in ACFS(G)$. Let $g_1, g_2 \in G$, then

$$\begin{aligned} f^{-1}(r_1)(g_1g_2g_1^{-1}) &= r_1(f(g_1g_2g_1^{-1})) = r_1(f(g_1)f(g_2)f(g_1^{-1})) = r_1(f(g_1)f(g_2)f^{-1}(g_1)) \\ &\leq S(r_1(f(g_2)), r_2(f(g_1))) = S(f^{-1}(r_1)(g_2), f^{-1}(r_2)(g_1). \end{aligned}$$

Also

$$\begin{aligned} f^{-1}(w_1)(g_1g_2g_1^{-1}) &= w_1(f(g_1g_2g_1^{-1})) = w_1(f(g_1)f(g_2)f(g_1^{-1})) = w_1(f(g_1)f(g_2)f^{-1}(g_1)) \\ &\leq \max\{w_1(f(g_2)), w_2(f(g_1))\} = \max\{f^{-1}(w_1)(g_2), f^{-1}(w_2)(g_1\}. \end{aligned}$$

Therefore $f^{-1}(\mu_1) \bowtie f^{-1}(\mu_2)$. □

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