

On the KdV Equation with Hysteresis

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Abstract

This paper discusses the generalized play hysteresis operator in connection with the KdV equation. Results from the nonlinear semigroup theory are applied to assure the existence and uniqueness. The KdV equation with hysteresis is reduced to a system of differential inclusions and solved.

Keywords: Hysteresis Operator, KdV Equations with Hysteresis

1. Introduction

The word *hysteresis* originates in the Greek word *hysterein*, which is translated as *to be behind* or *to come later*. The related Greek word *hysteresis* means *shortcoming* or *lag in arrival*. Ewing in 1885 [1] defined hysteresis as follows: *When there are two quantities M and N such that cyclic variations of N cause cyclic variations of M, then if the changes of M lag behind those of N, we may say that there is hysteresis in the relation of M and N*. This definition gives an idea of what hysteresis is. The hysteresis is coupling to PDEs with hysteresis, which arise in many fields such plasticity, dynamics with friction, ferromagnetism, ferroelectricity, superconductivity, adsorption and desorption, biology, chemistry and economics.

We note that the phenomenon is similar to the standard approach within continuum mechanics related to the sixth Hilbert problem [2]. Hilbert's sixth problem is to axiomatize those branches of science in which mathematics is prevalent. It occurs on the list of Hilbert's problems given out in 1900. The explicit statement is the Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: *To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics*.

In the 1970s, Krasnoselskii and Pokrovskii studied the concept of *hysteresis operator*, acting in spaces of time dependent functions [2]. Further researches were developed in a series of monographies dedicated to the hysteresis in connection with PDEs and applicative prob-

lems [3-5]. A useful survey can be found in [6]. Nonlinear semigroup theory in a Hilbert space was developed by Kōmura [7] and extended to Banach spaces by Crandall and Liggett [8] and Barbu [9]. Nonlinear semigroup theory represents a widely used tool for solving nonlinear PDEs. A survey of basic relevant results from a nonlinear semigroup theory, formulated generally in a Banach space is presented in [10,11].

Several models of mechanical and magnetic hysteresis may be represented via analogical models, namely the rheological models in mechanics, circuital models in electromagnetism, by arranging elementary components in series and/or in parallel [12-14]. These models consist of a family of elements, which can be interpreted as representing the mesoscopic structure of a composite material. Therefore, the procedure known as homogenization may be applied to provide an averaged representation of the system [15].

In this paper, the generalized play operator is analyzed in connection with the KdV equation. The problem is reduced to a system of differential inclusions and solved. This work is in the framework of the Visintin researches on models of hysteresis phenomena and on related PDEs [5,6,16-19].

2. Hysteresis Operators

In order to simplify the meaning of the hysteresis, let us consider a system whose the state is characterized by two scalar variables, the input function $u(t)$ and the output function $w(t)$, confined to a set $L \subset \mathbb{R}^2$. $\forall t \in [0, T]$. The function $w(t)$ depends on the previous evolution of $u(t)$ (memory effect) and on the initial state w_0 ,

such as

$$\begin{aligned} w(t) &= A(u, w_0)(t), \quad \forall t \in [0, T], \\ (u(0), w_0) &\in L, \quad A(u, w_0)(0) = w_0, \end{aligned} \tag{1}$$

where $A(u, w_0)$ is a memory operator defined in a Banach space of time-dependent functions for any fixed w_0 . The memory operator is causal: for $\forall (u_1, w_0), (u_2, w_0)$ with $u_1 = u_2$ in $[0, T]$, then $A(u_1, w_0)(t) = A(u_2, w_0)(t)$. Most typical hysteresis phenomena exhibit not purely rate-independent memory and as consequence, the rate-dependent effects are superposed to hysteresis. In the memory rate-dependent case, the *hysteresis operator* is not invariant with reference to any increasing diffeomorphism $\varphi: [0, T] \rightarrow [0, T]$, i.e. $A(u \circ \varphi, w_0) \neq A(u, w_0) \circ \varphi, \quad \forall t \in [0, T]$.

In the following we present the generalized play operator $w := A(u, w_0): R^+ \rightarrow R$ defined in the sense of Visintin (**Figure 1**). Let $u(t)$ be any continuous, piecewise linear function on R^+ , linear on $[t_{i-1}, t_i]$, $i = 1, 2, \dots$. We define $w(t) = A(u, w_0)(t)$ by

$$\begin{aligned} w(t) &= \min\{\gamma_l(u(0)), \max\{\gamma_r(u(0)), w_0\}\} \\ &\text{for } t = 0 \text{ and } w_0 \in R, \\ w(t) &= \min\{\gamma_l(u(t_i)), \max\{\gamma_r(u(t_i)), w(t_{i-1})\}\} \\ &\text{for } t \in (t_{i-1}, t_i), \quad i = 1, 2, \dots, \end{aligned} \tag{2}$$

where $\gamma_l, \gamma_r: R \rightarrow R$ are maximal monotone, possible multivalued functions with

$$\inf \gamma_r(u) \leq \sup \gamma_l(u), \quad \forall u \in R \tag{3}$$

Note that $w(0) = w_0$ only if $\gamma_r(u(0)) \leq w_0 \leq \gamma_l(u(0))$. The classical play operator can be obtained from the general play operator by choosing

$$\gamma_l(u) = u + r, \quad \gamma_r(u) = u - r, \tag{4}$$

with $r \geq 0$ a parameter, $u(t)$ a continuous input function on $[0, T]$ and $w_{r,0} \in [-r, r]$ an initial state. **Figure 2** presents the play operator with threshold r .

The hysteresis relationship with the PDEs can be written as [10].

$$w(x, t) = [A(u(x, \dots), w_0(x))](t) \text{ in } Q = \Omega \times [0, T], \tag{5}$$

where Ω is a bounded subset of R^n . The generalized play operator discussed in this paper is dissipative, in the sense that $\|(\lambda I - A)x\| \geq \lambda \|x\|$ for $\forall \lambda > 0$, where I is the identity mapping.

The PDEs with hysteresis can be transformed into systems of differential inclusions. The generalized play operator can be defined as a solution in the Sobolev space $W^{1,1}(0, T)$, $w \in W^{1,1}(0, T)$ of a variational inclusion of the type.

$$w_{,t} \in \phi(u, w) \text{ in } (0, T), \quad w(0) = w_0. \tag{6}$$

The norm in $W^{1,1}(0, T)$ is defined as

$$\|f\|_{k,p} = \left(\sum_{i=0}^k \|f^{(i)}\|_p^p \right)^{1/p} = \left(\sum_{i=0}^k \int |f^{(i)}|^p dt \right)^{1/p}.$$

The rate-independent differential inclusion is

$$\begin{aligned} w_{,t} &\in \phi(u, w) \\ &= \begin{cases} \{\infty\} & \text{if } w < \inf \gamma_r(u), \\ [0, +\infty] & \text{if } w \in \gamma_r(u)/\gamma_l(u), \\ \{0\} & \text{if } \sup \gamma_r(u) < w < \inf \gamma_l(u), \\ [-\infty, 0] & \text{if } w \in \gamma_l(u)/\gamma_r(u), \\ \{-\infty\} & \text{if } w > \sup \gamma_r(u), \\ [-\infty, +\infty] & \text{if } w \in \gamma_l(u) \cap \gamma_r(u). \end{cases} \end{aligned} \tag{7}$$

If γ_r and γ_l are Lipschitz-continuous, then the generalized play operator transforms $(u, v) \in W^{1,1}(0, T) \times R$ into the unique function $w \in W^{1,1}(0, T)$ such that $w(0)$ is the projection of v into $[\gamma_r(u(0)), \gamma_l(u(0))]$ and (7) is satisfied. The operator can be extended to $C^0([0, T]) \times R$, and it is equivalent to a variational inequality [20].

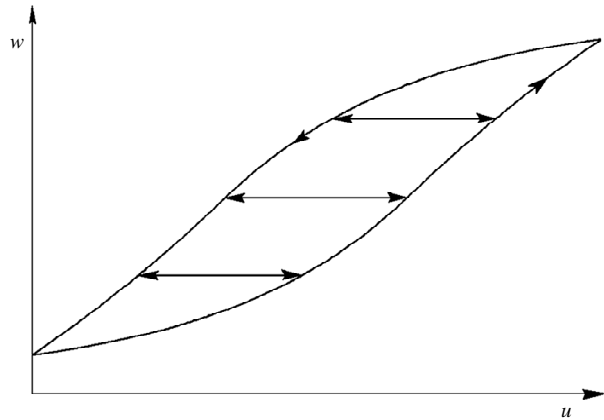


Figure 1. The generalized play operator.

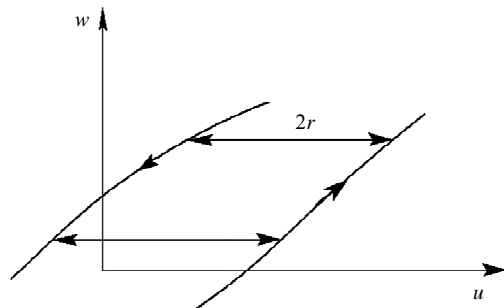


Figure 2. The play operator with threshold r .

We present here one example of PDE with hysteresis [10]

$$(u + w)_{,t} - \Delta u = f \text{ in } Q, \tag{8}$$

related to a generalized play operator (3) by (6), is formally equivalent to [5,10]

$$u_{,t} + \xi - \Delta u = f, \quad w_{,t} - \xi = 0, \quad \xi \in \phi(u, w) \text{ in } Q, \tag{9}$$

where ϕ is defined by (7) and comma represents the differentiation with respect to the specified variable. The Cauchy problem for (9) coupled with homogeneous Dirichlet boundary conditions as

$$F \in U_{,t} + A_1 U \text{ in } Q, \quad U(0) = U_0 \text{ in } \Omega, \tag{10}$$

where

$$\begin{aligned} U &= (u, w)^T, \quad F = (f, 0)^T, \\ A_1 U &= (\xi - \Delta u, -\xi)^T, \quad \xi \in \phi(U) \cap R. \end{aligned} \tag{11}$$

At the end of this section some results of the nonlinear semigroup theory are presented in the spirit of [10]. Let B be a Banach space, A nonlinear and multivalued hysteresis operator $A : D(A) \subset B \rightarrow B$ is accretive if

$$\|u_1 - u_2\|_B \leq \|u_1 - u_2 + \lambda(v_1 - v_2)\|_B, \quad \forall \lambda > 0. \tag{12}$$

Definition 2. Let B be a Banach space, the hysteresis operator A is called m -accretive if $Rg(I + \lambda A) = B$, $\forall \lambda > 0$.

Suppose that the derivative in the evolution equation can be approximated by a backward-difference quotient of step size $h > 0$ and f by a step functions f_k^h . We have

$$f_k^h \in \frac{u_k^h - u_{k-1}^h}{h} + A(u_k^h), \quad k = 1, 2, \dots, \quad u_0^h = u_0, \tag{13}$$

$$u_k^h(t) = u_k^h, \quad \text{for } kh \leq t < (k+1)h. \tag{14}$$

The scheme (13) is uniquely solved recursively and the Crandall-Liggett theorem holds:

Theorem 1. (Crandall-Liggett) [8]: If A is m -accretive, $f \in L^1(0, T, B)$ and $u_0 \in \bar{D}(A)$ and $f^h \rightarrow f$ in $L^1(0, T, B)$, then $u^h \rightarrow u$ uniformly as $h \rightarrow 0$ and $u \in C(0, T, B)$.

Theorem 2: If A is m -accretive, $f \in L^1(0, T, B)$ and $u_0 \in \bar{D}(A)$, then the Cauchy problem

$$f \in u_{,t} + A(u(t)), \quad u(0) = u_0. \tag{15}$$

has one and only one integral solution u . For $f = 0$, we have $u = S(t)u_0$, where $S(t)$ is a nonlinear semi-group of contractions generated by the operator A . If f has bounded variation in $[0, T]$ and $u_0 \in D(A)$, then the integral solution is Lipschitz continuous.

Definition 3. The function u is an integral solution of (15) in the sense of Benilan if 1) $u : [0, T] \rightarrow B$ is continuous; 2) $u(t) \in \bar{D}(A)$ for any $t \in [0, T]$; and 3)

$$u(0) = u_0 \text{ and}$$

$$\begin{aligned} \|u(t_2 - v)\|_B^2 &\leq \|u(t_1 - v)\|_B^2 \\ + 2 \int_{t_1}^{t_2} \lim_{\lambda \rightarrow 0} \frac{\|u(\tau - v) + \lambda(f(\tau) - z)\|_B^2 - \|u(\tau - v)\|_B^2}{2\lambda} d\tau. \end{aligned} \tag{16}$$

3. The KdV Equation with Hysteresis

Amplitude equations governing the non-linear resonant interaction of equatorial baroclinic and barotropic Rossby waves were derived by Majda and Biello [21,22] and used as a model for long range interactions between the tropical and mid-latitude troposphere. Exploiting the fact that some of the Rossby waves can resonantly interact, Majda and Biello [23] developed a small amplitude theory of nearly dispersionless, long equatorial Rossby waves. The analytic solitary wave solutions can be constructed with the functional form of the KdV soliton. These results inspire us to analyse the KdV equation from the point of view of the hysteresis of waves.

The KdV equation with hysteresis can be written under the form

$$\begin{aligned} q_{,t} &= q_{,xxx} - 6q(x, t)q_{,x} \\ \text{in } Q &= [-\infty, \infty] \times [0, T], \\ q(x, t) &= u(x, t) + w(x, t), \\ \lim_{x \rightarrow \pm\infty} u(x, t) &= 0, \\ w(x, t) &= A(u(x, t), w_0), \quad w_{,t} \in \phi(u, w), \\ w(x, 0) &= w_0(x) \text{ in } (0, T), \end{aligned} \tag{17}$$

where $\phi(u, w)$ is defined by (7). The hysteresis relation (5) representing a generalized play is also valid.

For $w = 0$, the exact solution of (17) is obtained by choosing the solution under the form $u(x, t) = z(x - \beta t)$. The exact solution is a solitary wave

$$u(x, t) = \frac{\beta}{2} \operatorname{sech}^2 \left[\frac{\sqrt{\beta}}{2} (x - \beta t) \right]$$

[24]. In order to have a real solution the quantity β must be a positive number. For $\beta > 0$ the solitary wave moves to the right, and the amplitude of the solitary wave is proportional to the speed which is indicated by the value of β . Thus larger amplitude solitary waves move with a higher speed than smaller amplitude waves.

To solve (17) we use the Lax formalism [25]. Equation (17) can be described by two operators depending of the hysteresis operator $A = A(u(x, t), w_0)$

$$L(A) = -\partial_{xx} + q, \quad M(A) = -4\partial_{xxx} + 3(q\partial_x + \partial_x q). \tag{18}$$

The operator $L(A)$ characterizes the spectral problem

$$L(A)\psi(x,t,\lambda) = \lambda\psi(x,t,\lambda), \quad (19)$$

and the operator $M(A)$ characterizes the t -evolution of the wavefunction $\psi(x,t,\lambda)$

$$\psi_{,t}(x,t,\lambda) = -M(A)\psi(x,t,\lambda). \quad (20)$$

The compatibility of (19) and (20) when λ is not dependent of t implies the Lax equation $L_{,t} = [L, M]$.

The algebraic properties which derive from the existence of the operator $L(A)$ refer to the existence of a recursor operator $\Lambda(A)$, and the existence of Bäcklund and Darboux transformations [24,25].

Starting from (19), we can look for operators $M_j(A)$ such that to have satisfied

$$L_{,t} = [L, M_j]. \quad (21)$$

Consider the following set of operator equations

$$[L, M] = V(A), \quad [L, \tilde{M}] = \tilde{V}(A), \quad (22)$$

where V and \tilde{V} are scalar functions for the operators M and \tilde{M} . Taking account the structure of the operator $L(A)$, we have

$$\tilde{M}(A) = L(A)M(A) + F(A)\partial_x + G(A), \quad (23)$$

where F and G are scalar functions of A and its derivative and of V defined by (22).

From Equations (22) and (23) it results that \tilde{V} can be expressed as a recursor operator $\Lambda(A)$ on the function V and depending on the hysteresis operator A

$$\Lambda V(x,t) = -\frac{1}{4}V_{,xx}(x,t) + q(x,t)V(x,t) - \frac{1}{2}q_{,x}(x,t)\int_x^\infty V(y,t)dy. \quad (24)$$

By using (24), the problem (17) becomes

$$\begin{aligned} (u+w)_{,t} &= \Lambda(A)(u+w)_{,x} \text{ in } Q = [-\infty, \infty] \times [0, T], \\ \lim_{x \rightarrow \pm\infty} u(x,t) &= 0, \\ w &= A(u, w_0), \quad w_{,t} \in \phi(u, w), \\ w(x, 0) &= w_0(x) = \sin(kx) \text{ in } (0, T), \end{aligned} \quad (25)$$

where $\phi(u, w)$ is defined by (7). In the spirit of Visintin [5], the problem (25) is formally equivalent to a system of differential inclusions

$$\begin{aligned} u_{,t} + \xi &= \Lambda(A)(u+A)_{,x} \text{ in } Q = [-\infty, \infty] \times [0, T], \\ \lim_{x \rightarrow \pm\infty} u(x,t) &= 0, \\ w_{,t} - \xi &= 0, \quad \xi \in \phi(u, w), \\ w(x, 0) &= w_0(x) = \sin(kx) \text{ in } Q, \end{aligned} \quad (26)$$

where $\phi(u, w)$ is defined by (7).

Figure 3 illustrates the hysteretic solution of the pro-

blem (26) for $k=3$, $t \in [0, 7]$, $x \in [-40, 40]$. For $t < 6$ the curve is a helix, then the solution exhibits several hysteretic loops for $t \geq 6$. The transition from a helix into the hysteresis loops is greatly aided by the excitation history expressed as a superposition of solitary waves. The transition instantaneously occurs as in the case of the climatologically appropriate mean winds and shears.

For $t \geq 20$, an intriguing aspect of the interaction appears by splitting of the hysteresis loop into two distinctive branches. **Figure 4** presents these two branches for $t \in [20, 100]$, $x \in [-30, 30]$. The solution varies between two hysteresis branches depending of the excitation history. Such branches, the splitting and formation of a double-sides comblike hysteresis loops have been observed experimentally [26].

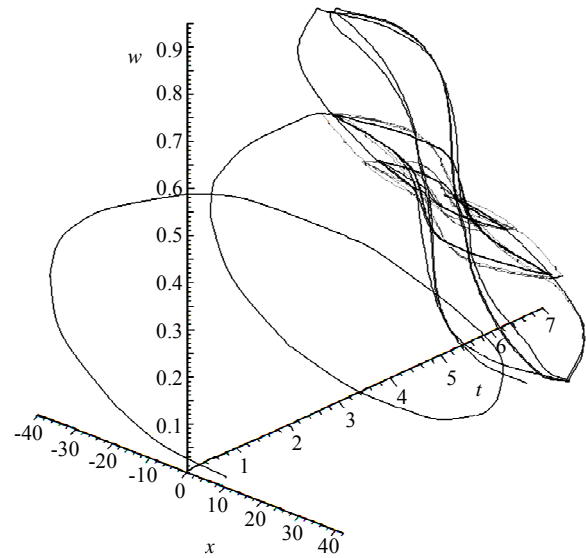


Figure 3. The hysteretic solution $w(x,t)$ of the problem (26) $t \in [0, 7]$.

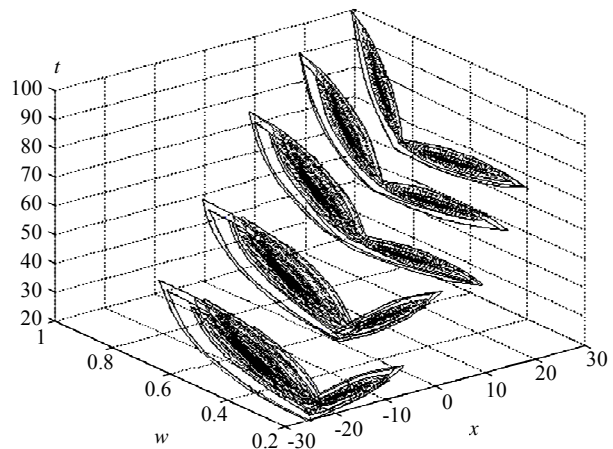


Figure 4. The hysteretic solution $w(x,t)$ of the problem (26) $t \in [20, 100]$.

4. Conclusions

This paper is aimed to outline some of the basic elements of the hysteresis operators in connection with PDEs. The construction of the KdV equation with hysteresis is just an example of a more general method developed by Visintin [5,6]. The KdV equation with hysteresis is reduced to a system of differential inclusions and solved.

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