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Numerical Treatment of Nonlinear Volterra-Fredholm Integral Equation with a Generalized Singular Kernel

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Abstract

In the paper, the approximate solution for the two-dimensional linear and nonlinear Volterra-Fredholm integral equation (V-FIE) with singular kernel by utilizing the combined Laplace-Adomian decomposition method (LADM) was studied. This technique is a convergent series from easily computable components. Four examples are exhibited, when the kernel takes Carleman and logarithmic forms. Numerical results uncover that the method is efficient and high accurate.

Keywords

Singular Integral Equation, Linear and Nonlinear V-FIE, Adomian Decomposition Method (ADM), Carleman Kernel, Logarithmic Kernel

1. Introduction

The V-FIE arises from parabolic boundary value problems. In practical applications one frequently encounters the V-FIE with singular kernel of the form

$$\phi(\chi,t) = \varsigma(\chi,t) + \lambda \int_0^t \int_\Omega F(t,\rho) k(|\chi-\eta|) \gamma(\eta,\rho,\phi(\eta,\rho)) \mathrm{d}\eta \mathrm{d}\rho \qquad (1)$$

The functions $k(|\chi-\eta|)$, $F(t,\rho)$ and $\varsigma(\chi,t)$ are given and called the kernel of Fredholm integral term, Volterra integral term and the free term respectively and λ is a real parameter (may be complex and has physical meaning). Also, Ω is the domain of integration with respect to position, and the time t, $\rho \in [0,\tilde{T}]$, $\tilde{T} < \infty$. While $\phi(\chi,t)$ is the unknown function to be determined in the space $L_p(\Omega) \times C[0,\tilde{T}]$. In [1] Abdou et al.

studied the existence and uniqueness of solution of V-FIE.

There are several techniques that have been utilized to handle the integral Equation (1), in [2]-[5] a few techniques, for example, the projection method, time collocation method, the trapezoidal Nystrom method, and furthermore analytical or numerical techniques were utilized to treated this equation, but this techniques experienced troubles as far as computational work utilized. In [6] treated Maleknejad and Hadizadeh Equation (1) by using the ADM presented in [7]-[9].

Many authors have studied solutions of two-dimensional linear and nonlinear integral equations by utilizing different techniques, such as Abdou *et al.* in [10] discussed the solution of linear and nonlinear Hammerstien integral equations with continuous kernel and used two different methods (Adomian decomposition method and homotopy analysis method). Abdou *et al.* in [11]-[13] considered the integral equation with singular kernel and used Toeplitz matrix and product Nystrom methods to obtain the solution. In [14] El-Kalla and Al-Bugami used ADM and degenerate kernel method for solving nonlinear V-FIE with continuous kernel.

In this paper, we will discuss the combined (LADM) to approximate solutions with high degree of accuracy for V-FIE with a generalized singular kernel.

2. The Adomian Decomposition Method for Solving V-FIE

Consider the integral equation

$$\phi(\chi,t)=\varsigma(\chi,t)+\lambda\int_0^t\int_\Omega F(t,\rho)k(|\chi-\eta|)\gamma(\eta,\rho,\phi(\eta,\rho))\mathrm{d}\eta\mathrm{d}\rho \tag{2}$$

The (ADM) introduces the following expression

$$\phi(\chi,t)=\sum_{i=0}^\infty \phi_i(\chi,t) \tag{3}$$

for the solution $\phi(\chi,t)$ of Equation (2), where the components $\phi_i(\chi,t)$ will be determined recurrently. Moreover, the method defines the nonlinear function $\gamma(\eta,\rho,\phi(\eta,\rho))$ by an infinite series of polynomials

$$\gamma(\eta,\rho,\phi(\eta,\rho))=\sum_{n=0}^\infty A_n \tag{4}$$

where A_n are the so-called Adomian polynomials that represent the nonlinear term $\gamma(\eta,\rho,\phi(\eta,\rho))$ and can be calculated for various classes of nonlinear operators according to specific algorithms set by Adomian [8] [9]. A new algorithm for calculating these polynomials was established by Wazwaz [15] [16].

Substituting Equation (3) and Equation (4) into Equation (2) yields

$$\sum_{i=0}^\infty \phi_i(\chi,t)=\varsigma(\chi,t)+\lambda\int_0^t\int_\Omega F(t,\rho)k(|\chi-\eta|)\sum_{i=0}^\infty A_i\mathrm{d}\eta\mathrm{d}\rho \tag{5}$$

The components $\phi_i(\chi,t),i\geq 0$ are computed using the following recursive relations

$$\phi_0(\chi,t)=\varsigma(\chi,t) \tag{6}$$

$$\phi_{i+1}(\chi,t)=\lambda\int_0^t\int_\Omega F(t,\rho)k(|\chi-\eta|)A_i\mathrm{d}\eta\mathrm{d}\rho \tag{7}$$

Relations (6,7) will enable us to determine the components $\phi_i(\chi,t),i\geq 0$ recurrently, and as a result, the series solution of $\phi(\chi,t)$ is readily obtained.

3. Laplace Adomian Decomposition Method Applied to V-FlE with Singular Kernel

3.1. Carleman Kernel

We assume that the kernel $k(|\chi-\eta|)$ of Equation (1) takes the form $k(|\chi-\eta|)=|\chi-\eta|^{-\nu}$, [17] then integral Equation (1) can be expressed as:

$$\phi(\chi,t)=\varsigma(\chi,t)+\lambda\int_0^t\int_\Omega F(t,\rho)|\chi-\eta|^{-\nu}\gamma(\eta,\rho,\phi(\eta,\rho))\mathrm{d}\eta\mathrm{d}\rho \tag{8}$$

Applying the Laplace transform to both sides of Equation (8) gives:

$$\mathcal{L}\{\phi(\chi,t)\} = \mathcal{L}\{\varsigma(\chi,t)\} + \lambda\mathcal{L}\left[\{F(t,\rho)\}\{|\chi-\eta|^{-\nu}\}\{\gamma(\eta,\rho,\phi(\eta,\rho))\}\right] \tag{9}$$

The ADM can be used to handle Equation (9). We represent the linear term $\phi(\chi,t)$ from Equation (3) and the nonlinear term $\gamma(\eta,\rho,\phi(\eta,\rho))$ will be represented by the Adomian polynomials from Equation (4).

Substituting Equation (3) and Equation (4) into Equation (9) leads to

$$\mathcal{L}\left\{\sum_{i=0}^{\infty}\phi_i(\chi,t)\right\} = \mathcal{L}\{\varsigma(\chi,t)\} + \lambda\mathcal{L}\left[\{F(t,\rho)\}\{|\chi-\eta|^{-\nu}\}\left\{\sum_{i=0}^{\infty}A_i\right\}\right] \tag{10}$$

The ADM introduces the recursive relation

$$\mathcal{L}\{\phi_0(\chi,t)\} = \mathcal{L}\{\varsigma(\chi,t)\}$$

$$\mathcal{L}\{\phi_{i+1}(\chi,t)\} = \lambda\mathcal{L}\left[\{F(t,\rho)\}\{|\chi-\eta|^{-\nu}\}\{A_i\}\right], \ i\geq 0 \tag{11}$$

Applying the inverse Laplace transform to the first part of Equation (11) gives $\phi_0(\chi,t)$. Utilizing $\phi_0(\chi,t)$ will empower us to evaluate $\phi_1(\chi,t)$, and so on. This will prompt the complete determination of the components of $\phi_i, i\geq 0$ upon utilizing the second part of Equation (11). The series solution follows promptly after utilizing Equation (3). The obtained series solution may converge to an exact solution if such a solution exists.

3.2. Logarithmic Kernel

We assume that the kernel $k(|\chi-\eta|)$ of Equation (1) takes the form $k(|\chi-\eta|) = \ln|\chi-\eta|$, [17] then integral Equation (1) can be expressed as:

$$\phi(\chi,t) = \varsigma(\chi,t) + \lambda\int_0^t\int_\Omega F(t,\rho)\ln|\chi-\eta|\gamma(\eta,\rho,\phi(\eta,\rho))\,d\eta\,d\rho \tag{12}$$

Applying the Laplace transform to both sides of Equation (12) gives:

$$\mathcal{L}\{\phi(\chi,t)\} = \mathcal{L}\{\varsigma(\chi,t)\} + \lambda\mathcal{L}\left[\{F(t,\rho)\}\{\ln|\chi-\eta|\}\{\gamma(\eta,\rho,\phi(\eta,\rho))\}\right] \tag{13}$$

Using the same method we shall find at the end the required solution by the inverse of Laplace transform.

4. Numerical Examples

4.1. Application for Carleman Kernel and Logarithmic Kernel

We consider two examples for the integral equation

$$\phi(\chi,t) = \varsigma(\chi,t) + \lambda\int_0^t\int_{-1}^1\rho^2 k(|\chi-\eta|)\phi^l(\eta,\rho)\,d\eta\,d\rho \tag{14}$$

$\lambda = 0.001, N = 20$, the exact solution $\phi(\chi,t) = \chi^2 t^2$

We consider the linear and nonlinear cases: $l = 1, l = 0.5$ respectively, for the Carleman kernel $k(|\chi-\eta|) = |\chi-\eta|^{-\nu}$ and the computing results are obtained when $\nu = 0.01, 0.32$ where ν is called Poisson's coefficient, $0 < \nu < 1$, while the kernel in the second example takes the logarithmic kernel $k(|\chi-\eta|) = \ln|\chi-\eta|$ and the results are computing, using Maple 17 at $t = 0.001, t = 0.7$ and $N = 20$.

Example 1 [(see 1)]: Consider the V-FIE with Carleman kernel

$$\phi(\chi,t) = \varsigma(\chi,t) + \lambda\int_0^t\int_{-1}^1\rho^2|\chi-\eta|^{-\nu}\phi^l(\eta,\rho)\,d\eta\,d\rho \tag{15}$$

$\lambda = 0.001, N = 20$, the exact solution $\phi(\chi,t) = \chi^2 t^2$

Using Maple 17, we obtain **Table 1** and **Table 2**

Example 2 [(see 1)]: Consider the V-FIE with logarithmic kernel

$$\phi(\chi,t) = \varsigma(\chi,t) + \lambda\int_0^t\int_{-1}^1\rho^2\ln|\chi-\eta|\phi^l(\eta,\rho)\,d\eta\,d\rho \tag{16}$$

$\lambda = 0.001, N = 20$, the exact solution $\phi(\chi,t) = \chi^2 t^2$

Using Maple 17, we obtain **Table 3** and **Table 4**

Table 1. Results obtained for example 1 and error (Linear case, $l=1$).

t	χ	Exact	App.	Error	App.	Error
			$v = 0.01$		$v = 0.32$	
	−1.00E+00	1.000000000E−06	1.00000000E−06	0.000000000E+00	1.00000000E−06	0.000000000E+00
$t = 0.001$	−5.00E−01	2.50000000E−07	2.500000000E−07	0.000000000E+00	2.500000000E−07	0.000000000E+00
	5.00E−01	2.500000000E−07	2.500000000E−07	0.000000000E+00	2.500000000E−07	0.000000000E+00
	1.00E+00	1.000000000E−06	9.999999997E−07	3.000000000E−16	9.999999997E−07	3.000000000E−16
			$v = 0.01$		$v = 0.32$	
	−1.00E+00	4.900000000E−01	4.900814897E−01	8.148970000E−05	4.900840230E−01	8.402300000E−05
$t = 0.7$	−5.00E−01	1.225000000E−01	1.225102564E−01	1.025640000E−05	1.225131121E−01	1.31121000E−05
	5.00E−01	1.225000000E−01	1.224897398E−01	1.026020000E−05	1.224755300E−01	2.447000000E−05
	1.00E+00	4.900000000E−01	4.899184905E−01	8.150950000E−05	4.898431989E−01	1.568011000E−04

Table 2. Results obtained for example 1 and error (Nonlinear case, $l=0.5$).

t	χ	Exact	App.	Error	App.	Error
			$v = 0.01$		$v = 0.32$	
	−1.00E+00	1.000000000E−06	9.999994927E−07	5.073000000E−13	9.999995310E−07	4.690000000E−13
$t = 0.001$	−5.00E−01	2.50000000E−07	2.499998723E−07	1.277000000E−13	2.499998536E−07	1.464000000E−13
	5.00E−01	2.500000000E−07	2.499998722E−07	1.278000000E−13	2.499997268E−07	2.732000000E−13
	1.00E+00	1.000000000E−06	9.999994924E−07	5.076000000E−13	9.999991246E−07	8.754000000E−13
			$v = 0.01$		$v = 0.32$	
	−1.00E+00	4.900000000E−01	4.898260141E−01	1.739859000E−04	4.898390761E−01	1.609239000E−04
$t = 0.7$	−5.00E−01	1.225000000E−01	1.224562011E−01	4.379890000E−05	1.224497750E−01	5.022500000E−05
	5.00E−01	1.225000000E−01	1.224561794E−01	4.38206000E−05	1.224063394E−01	9.366060000E−05
	1.00E+00	4.900000000E−01	4.898259283E−01	1.740717000E−04	4.896998594E−01	3.001406000E−04

Table 3. Results obtained for example 2 and error (Linear case, $l=1$).

t	χ	Exact	App.	Error
	−1.00E+00	1.000000000E−06	9.999999994E−07	6.000000000E−16
$t = 0.001$	−5.00E−01	2.50000000E−07	2.499999999E−07	1.000000000E−16
	5.00E−01	2.500000000E−07	2.500000001E−07	1.000000000E−16
	1.00E+00	1.000000000E−06	1.000000001E−06	1.000000000E−15
	−1.00E+00	4.900000000E−01	4.898532159E−01	1.467841000E−04
$t = 0.7$	−5.00E−01	1.225000000E−01	1.224747231E−01	2.527690000E−05
	5.00E−01	1.225000000E−01	1.225252799E−01	2.527990000E−05
	1.00E+00	4.900000000E−01	4.901467682E−01	1.467682000E−04

4.2. Application for a Generalized Carleman Kernel and Logarithmic Kernel

Example 3 [(see 11, 13)]: Consider the V-FIE with generalized Carleman kernel

$$\phi(\chi,t) = \varsigma(\chi,t) + \lambda \int_0^t \int_{-1}^1 \rho^2 \left|\chi^4 - \eta^4\right|^{-v} \phi^l(\eta,\rho)\,d\eta d\rho, \quad \left(0 \le t \le \tilde{T}; |\chi| \le 1\right) \tag{17}$$

$\lambda = 0.31579, v = 0.12, N = 20$, the exact solution $u(x,t) = x^5 t^6$
Using Maple 17, we obtain **Table 5**.

Example 4 [(see 11,13)]: Consider the V-FIE with generalized logarithmic kernel

$$\phi(\chi,t) = \varsigma(\chi,t) + \lambda \int_0^t \int_{-1}^1 \rho^2 \ln\left|\chi^4 - \eta^4\right| \phi^l(\eta,\rho) \,d\eta d\rho \qquad (18)$$

$\lambda = 0.6666666667, N = 20$, the exact solution $u(x,t) = x^5 t^6$,
Using Maple 17, we obtain **Table 6**.

Table 4. Results obtained for example 2 and error (Nonlinear case, $l = 0.5$).

t	χ	Exact	App.	Error
	−1.00E+00	1.000000000E−06	1.000000750E−06	7.500000000E−13
	−5.00E−01	2.50000000E−07	2.500002741E−07	2.741000000E−13
$t = 0.001$	5.00E−01	2.500000000E−07	2.500002741E−07	2.741000000E−13
	1.00E+00	1.000000000E−06	1.000000750E−06	7.500000000E−13
	−1.00E+00	4.900000000E−01	4.902570318E−01	2.2570318000E−04
	−5.00E−01	1.225000000E−01	1.225939984E−01	9.399840000E−05
$t = 0.7$	5.00E−01	1.225000000E−01	1.225940725E−01	9.407250000E−05
	1.00E+00	4.900000000E−01	4.902573282E−01	2.573282000E−04

Table 5. Results obtained for example 3 and error.

t	χ	Exact	Linear $(l = 1)$		Nonlinear $(l = 2)$	
			App.	Error	App.	Error
	−1.00E+00	−4.665600000E−14	−4.665602285E−14	2.285000000E−20	−4.665600000E−14	0.000000000E+00
	−5.00E−01	−1.45800000E−15	−1.458000498E−15	4.980000000E−22	−1.45800000E−15	0.000000000E+00
$t = 0.006$	5.00E−01	1.45800000E−15	1.457992071E−15	7.929000000E−21	1.45800000E−15	0.000000000E+00
	1.00E+00	4.665600000E−14	4.665563615E−14	3.638500000E−19	4.665600000E−14	0.000000000E+00
	−1.00E+00	−4.096000000E−03	−4.100223060E−03	4.223060000E−06	−4.095973539E−03	2.646100000E−08
	−5.00E−01	−1.280000000E−04	−1.281230657E−04	1.230657000E−07	−1.279999820E−04	1.800000000E−11
$t = 0.4$	5.00E−01	1.280000000E−04	1.249760487E−04	3.023951300E−06	1.279997138E−04	2.862000000E−10
	1.00E+00	4.096000000E−03	3.958591126E−03	1.374088740E−04	4.095579754E−03	4.202460000E−07

Table 6. Results obtained for example 4 and error.

t	χ	Exact	Linear $(l = 1)$		Nonlinear $(l = 2)$	
			App.	Error	App.	Error
	−1.00E+00	−4.665600000E−14	−4.665417105E−14	1.828950000E−18	−4.665600000E−14	0.000000000E+00
	−5.00E−01	−1.45800000E−15	−1.457963338E−15	3.666200000E−20	−1.45800000E−15	0.000000000E+00
$t = 0.006$	5.00E−01	1.45800000E−15	1.458036663E−15	3.666300000E−20	1.45800000E−15	0.000000000E+00
	1.00E+00	4.665600000E−14	4.665782899E−14	1.828990000E−18	4.665600000E−14	0.000000000E+00
	−1.00E+00	−4.096000000E−03	−3.369778189E−03	7.262218110E−04	−4.097965280E−03	1.965280000E−06
	−5.00E−01	−1.280000000E−04	−1.139984028E−04	1.400159720E−05	−1.280011799E−04	1.179900000E−09
$t = 0.4$	5.00E−01	1.280000000E−04	1.439401751E−04	1.594017510E−05	1.280011799E−04	1.179900000E−09
	1.00E+00	4.096000000E−03	4.947706301E−03	8.51706301E−04	4.097966603E−03	1.966603000E−06

5. Conclusion

In this study, we considered linear and nonlinear integral equations of type Volterra-Fredholm with singular kernel. We have proven that the (LADM) is effective and useful technique for solving these kinds of integral equations with singular kernel and many nonlinear problems, efficiency and accuracy of the introduced method are illustrated by four numerical examples which showed simplicity of this method.

Acknowledgements

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