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An Application of Mathematical Methods for Solving of Scientific Problems

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Authors' contributions

This work was carried out in collaboration between all authors. All of authors designed, prepared and approved the final manuscript.

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ABSTRACT

As is known, the formation and development of the mathematical methods are largely related with the successful application of these methods to solve problems of the natural sciences. Recently the area of application of the mathematical methods is greatly expanded. It is difficult to find an area where it is not used in solving different problems. As is known, is appeared the necessity to solve the differential, integral and integro-differential equations in solving of scientific problems. Note that one of the basic questions in the modern computational mathematics consists of construction of the numerical methods with the high accuracy. Therefore, here, is proposed the multi-step method with higher accuracy than the known. And is constructed the stable methods with the expanded area of stability for solving the above mentioned equations.

Keywords: Initial-value problem for ODE; Volterra integral equation; multistep multiderivative methods (MMM); hybrid methods; initial-value problem for integro-differential equations.

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1. INTRODUCTION

As is known, there are some arsenals of methods which can apply for solving of the differential, integral and integro-differential equations. Among these methods, there are methods which apply both to solving of integral and the integro-differential equations. Here, also proposed one multi-step method with high accuracy, which is applied to solving the ordinary differential equations, integral and integro-differential equations of Volterra type.

If we take into account the chronology of investigations of these equations, we find that the most common and earliest investigated are the ordinary differential equations, which in its general form we can write as follows:

$$y' = f(x, y), \ \ x \in [x_0, X]. \tag{1}$$

One of the most common problems for the equation (1) is the initial value problem. Therefore, consider finding of such solution of equation (1) which satisfies the condition:

$$y(x_0) = y_0 . \tag{2}$$

Many papers dedicated to investigation of the problem (1)-(2) (e.g. [1-27]).

We assume that the problem (1) - (2) has a unique solution $y(x)$, which was defined on the interval $[x_0, X]$. The segment $[x_0, X]$ divided into N equal parts by using constant step size $h > 0$ and the mesh points are denoted through $x_i = x_0 + ih \ (i = 0,1,...,N)$ for finding the numerical solution of the problem (1) and (2).

The approximate values of the solution of the problems (1) and (2) are denoted by $y_i \ (i = 0,1,..., N)$ at the points $x_i \ (i = 0,1,..., N)$, and the exact values are denoted through $y(x_i)(i = 0,1,..., N)$. The first paragraph is dedicated no investigation of the problem (1) - (2). We investigated the following Volterra integral equation in the second paragraph:

$$y(x) = f(x) + \int_{x_0}^{x} K(x, s, y(s))ds, x_0 \le s \le x \le X . \tag{3}$$

Note that the equation (3) is a Volterra integral equation of the second kind. Series of papers are dedicated to investigation of them (e.g. [28-44]). In the linear case, equation (3) was fundamentally investigated by Vito Volterra (e.g. [21],[22]). Significant part of these papers is dedicated to finding the numerical solution of the equation (3). Here, we propose some modification of the quadrature method (e.g. [25]–[30]). Usually we suggest that the investigated equation has a unique solution in the considered closed field for the study of the approximate solutions of some equations. Our work is no exception. Therefore, we assume that the equation (3) has a unique solution, defined on the interval $[x_0, X]$. In order to find the numerical solution of the equation (3), the segment $[x_0, X]$ divided into N equal parts by using constant step size $h > 0$ and the mesh points is defined as the $x_{i+1} = x_i + h \ (i = 0,1,..., N-1)$ and is denoted approximate value of the solution of equation (3) at the point $x_i \ (0 \le i \le N)$ by $y_i \ (0 \le i \le N)$, and the exact value of the solution of equation (3) at the point $x_i \ (0 \le i \le N)$ by $y(x_i)(0 \le i \le N)$.

If we apply the method of quadrature to find of the solution of the equation (3), then we have:

$$y_i = f_i + h\sum_{j=0}^{i} a_j^{(i)} K(x_i, x_j, y_j), y(x_0) = f(x_0), (i = 1,2,.., N). \tag{4}$$

As follows from here, the volume of computational work increases with increasing values of the variable i. Here, we offer the method that relieved of the specified disadvantage and has a higher accuracy than the known.

The third paragraph dedicated to investigation of the numerical solution of the initial value problem for integro-differential equation of Volterra type. To this end consider finding a numerical solution of the following problem.

$$y' = F(x, y) + \lambda \int_{x_0}^{x} K(x, s, y(s))ds,\ y(x_0) = y_0,\ x_0 \le s \le x \le X. \quad (5)$$

Obviously, the differential problem (5) in the most general form we can represent as follows:

$$y' = F(x, y, v(x)), y(x_0) = y_0, x \in [x_0, X], \quad (6)$$

where

$$v(x) = \int_{x_0}^{x} K(x, s, y(s))ds.$$

Assuming, that a unique solution of problem (5) exists, we construct a method for finding its numerical solution.

There are a number of papers devoted the solution of the problem (5) (e.g. [28], [29], [31], [33], [45] – [60]).

As mentioned above, the purpose of our research is the using of finite difference method to solve the problem (1) - (2) and (5), and apply it to solving of the equation (3). However, we face with the solving of integro-differential equations in investigation of many important objects of natural science, such as nuclear energy, the study of diseases of the season, including influenza, research of memory of land, sustainability of biological systems and ecosystems, populations of some organisms, etc. Here, we constructed methods for the solving the problems (5) that have higher accuracy and allow us to use simple algorithms for its solving.

2. MATERIALS AND METHODS

2.1 On a Way for Construction of the Numerical Methods with High Accuracy for Solving of the Problem (1) - (2)

As mentioned above, there is a wide arsenal of the numerical methods for solving initial value problem for the ODE. To compare them, scientists have proposed various criteria including the stability, the order, the degree, the stability region and others.

Usually these methods we divide into two classes: one step and multistep methods. Known representatives of them are the Runge-Kutta and Adams methods. Recently, we have constructed method at junction of one-step and multi-step methods, as a result of which are appeared the two-step Runge-Kutta methods, hybrid methods, etc. Adams method was generalized in the follow form in the middle of the XX century:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} \ (n = 0,1,..., N-k) \quad (7)$$

where the coefficients $\alpha_i, \beta_i \ (i = 0,1,...,k)$ are some real numbers, and $\alpha_k \neq 0$, but $f_m = f(x_m, y_m) \ (m \geq 0)$. The method (7) is called a multistep or k - step method with constant coefficients in the scientific literature. Note that the concepts of stability for the method of (7) were determined in two forms: language ε and δ (e.g. [21]). The criterion was proposed for stability of the method (7), which defined by its roots of the characteristic polynomials (7) in the paper [22]. In the future, this criterion was used to determine the stability of the method (7) (e.g. [23]-[24]). It is necessary to determine the maximum values of the accuracy of stable methods such as (7) in the study of many applied problems, which has found its solution in the famous work of Dahlkvist (e. g. [25]-[26]). Such questions were investigated for $k \leq 10$ and $\beta_k = 0$ in [24]. Method (7) is researched when $\beta_k = 0$ for any k in the paper [27]. Dahlkvist take into account that the value k is the order of the difference equation (7). Therefore, for determine order of the accuracy of the method (7) is used term degree. In the paper [27], we proved that the connection between the order k and the degree p of the stable methods of type (7), can write as:

$$p \leq 2[k/2] + 2, \quad (8)$$

but there are stable methods with the degree $p \le 2[k/2]+2$ for any k. Sometimes inequality (8) is called the first barrier of Dalkvist. The researchers used different schemes to construct a more accurate method as the Richardson extrapolation, the Hamming method, linear combination of some multi-step methods, etc. The maximum value of increasing degree of these methods is found by using of some schemes in the work [16]. However, some authors have used of higher derivatives of function for constructing more accurate methods $f(x, y)$, which can be written in the most general form as:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{m} h^j \sum_{i=0}^{k} \beta^{(j)} y_{n+i}^{(j)}. \tag{9}$$

This method was investigated for $m = 2$ by many authors (e.g. [5]-[8]). But method for the value $m \ge 3$ was investigated in the paper [8].

Here, is proposed to use hybrid methods for solving the problem (1) - (2), which in one variant can be written as:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} (\beta_i y'_{n+i} + \hat{\beta}_i y'_{n+i+\nu_i}),\ (|\nu_i| < 1,\ i = 0,1,...,\ k). \tag{10}$$

The values of these coefficients and values of $\nu_i (i = 0,1,2,..., k)$ can be found by using some bounds on the coefficients of the method (10) and by solving the next nonlinear system of algebraic equations:

$$\sum_{i=0}^{k} \alpha_i = 0,\ \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k} \beta_i + \sum_{i=0}^{k} \gamma_i,$$
$$\sum_{i=0}^{k} \frac{i^p}{p!}\alpha_i = \sum_{i=0}^{k} \frac{i^{p-1}}{(p-1)!}\beta_i + \sum_{i=0}^{k} \frac{l_i^{p-1}}{(p-1)!}\gamma_i. \tag{11}$$

here $l_i = i + \nu_i\ (i = 0,1,2,...k)$.

On the first let us consider any bounds for the coefficients of the method (10).

A: The coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i\ (i = 0,1,2,..., k)$ are some real numbers, moreover $\alpha_k \ne 0$.
B: Characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i,\quad \delta(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i;\ \gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i}.$$

have no common multipliers different roots from the constant.

C: $\sigma(1) + \gamma(1) \ne 0$ and $p \ge 1$.

We can prove that there exist stable methods with the degree $p \le 3k + 2$ in the methods of type (10).

Let us consider special cases and set $k = 2$ for illustration the existents of the hybrid methods with the degree $p = 10$. Therefore, we have from (11):

$$
\begin{aligned}
&\beta_0 + \beta_1 + \beta_2 + \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 = 2a_2 + a_1, \\
&\beta_1 + 2\beta_2 + l_0\hat{\beta}_0 + l_1\hat{\beta}_1 + l_2\hat{\beta}_2 = \frac{1}{2}(2^2 a_2 + a_1), \\
&\beta_1 + 2^2\beta_2 + l_0^2\hat{\beta}_0 + l_1^2\hat{\beta}_1 + l_2^2\hat{\beta}_2 = \frac{1}{3}(2^3 a_2 + a_1), \\
&\beta_1 + 2^3\beta_3 + l_0^3\hat{\beta}_0 + l_1^3\hat{\beta}_1 + l_2^3\hat{\beta}_2 = \frac{1}{4}(2^4 a_2 + a_1), \\
&\beta_1 + 2^4\beta_4 + l_0^4\hat{\beta}_0 + l_1^4\hat{\beta}_1 + l_2^4\hat{\beta}_2 = \frac{1}{5}(2^5 a_2 + a_1), \\
&\beta_1 + 2^5\beta_5 + l_0^5\hat{\beta}_0 + l_1^5\hat{\beta}_1 + l_2^5\hat{\beta}_2 = \frac{1}{6}(2^6 a_2 + a_1), \\
&\beta_1 + 2^6\beta_6 + l_0^6\hat{\beta}_0 + l_1^6\hat{\beta}_1 + l_2^6\hat{\beta}_2 = \frac{1}{7}(2^7 a_2 + a_1), \\
&\beta_1 + 2^7\beta_7 + l_0^7\hat{\beta}_0 + l_1^7\hat{\beta}_1 + l_2^7\hat{\beta}_2 = \frac{1}{8}(2^8 a_2 + a_1), \\
&\beta_1 + 2^8\beta_8 + l_0^8\hat{\beta}_0 + l_1^8\hat{\beta}_1 + l_2^8\hat{\beta}_2 = \frac{1}{9}(2^9 a_2 + a_1).
\end{aligned} \tag{12}
$$

Obviously, one can construct different methods which having the different properties by using the solution of the system (12). But, here we will investigate convergence of these methods. Therefore, we consider construction of the stable methods of type (10) and put $k = 2$, $a_0 = -1$, $a_1 = 0$, $a_2 = 1$. Then we obtain some methods by solving of the system (12) and one of which is following:

$$
\begin{aligned}
y_{n+2} = y_n + h(64 f_{n+2} + 98 f_{n+1} + 18 f_n)/180 + \\
+ h(18 f_{n+l_2} + 98 f_{n+l_2} + 64 f_{n+l_0})/180,
\end{aligned} \tag{13}
$$

here $l_2 = 1 + \sqrt{21}/14$, $l_1 = 1$, $l_0 = 1 - \sqrt{21}/14$.

Method (13) is stable and has the degree $p = 8$. The values y_{n+l_0}, y_{n+l_2} and also the value of variable y_{n+1} should be known for application of the method (13) to solving of some problems. Let us change h by $h/2$. Then the quantity y_{n+2} is replaced by the y_{n+1}, and from the formula (13), the next one step method is follows:

$$
\begin{aligned}
y_{n+1} = y_n + h(64 f_{n+1} + 98 f_{n+\frac{1}{2}} + 18 f_n)/360 + \\
+ h(18 f_{n+l_2/2} + 98 f_{n+1/2} + 64 f_{n+l_0/2})/360
\end{aligned} . \tag{14}
$$

Scheme (14) is a hybrid method with the degree $p = 8$. Remark that notion of the degree is defined for the method (2) as follows:

Definition 1. For a sufficiently smooth function $y(x)$ the method (10) has the degree $p > 0$, if the following is held:

$$
\sum_{i=0}^{k} \alpha_i y(x + ih) - h\sum_{i=0}^{k} (\beta_i y'(x + ih) + \gamma_i y'(x + (i + \nu_i)h)) = O(h^{p+1}),\ h \to 0,
$$

here $x = x_0 + nh$ is a fixed point, the variable p is integer.

Remark that stability of the method (10) is defined by the following way:

Definition 2. Method (10) is stable if all the roots of the polynomial

$$\rho(\lambda) = \alpha_{k-m}\lambda^{k-m} + \alpha_{k-m-1}\lambda^{k-m-1} + ... + \alpha_1\lambda + \alpha_0$$

lie inside the unit circle on the boundary of which does not have multiple roots.

Now consider the application of the next hybrid method of type (10)

$$y_{n+1} = y_n + h(f_{n+l_0} + f_{n+1+l_1})/2, \quad (l_1 = 1/2 + \sqrt{3}/6; \; l_0 = 1/2 - \sqrt{3}/6)$$

to solving following example:

$y' = \cos x, \; y(0) = 0, \; x \in [0,1].$ Exact solution: $y(x) = \sin x$.

The Simpson's method and the following method may be used for the calculation of the values y_{n+1}:

$$y_{n+1} = y_n + h(y'_n + 4y'_{n+1/2} + y'_{n+1})/6 \text{ . (*)}$$

The results of calculations accommodated in Table 1.

Table 1. Comparison of errors for stepsize $h = 0.1$

Step size	Variable x	Error for the Simpson's method	Error for the method (*)
$h = 0.1$	0.2	$0.11E-06$	$0.69E-08$
	0.3	$0.10E-06$	$0.10E-07$
	0.4	$0.21E-06$	$0.13E-07$
	0.5	$0.21E-06$	$0.16E-07$
	0.6	$0.31E-06$	$0.19E-07$
	0.7	$0.30E-06$	$0.22E-07$
	0.8	$0.39E-06$	$0.24E-07$
	0.9	$0.38E-06$	$0.27E-07$
	1.0	$0.46E-06$	$0.29E-07$

Scientists have suggested using the hybrid methods for construction of more accurate methods, which constructed at junction of Adams and Runge-Kutta methods.

2.2 Application of Some Finite-difference Method for Solving Nonlinear Volterra Integral Equations

The integral equations with the variable boundaries have wide application in solution of many problems related with different branches of natural sciences such as the study of atomic energy, some types of the earthquakes, some problems of communication, environmental and biological systems, weather, etc. Therefore, lately is increased interest in the investigation of the equation (3). There are necessity construct new methods with improved properties. Here, we show that there are such methods in class of finite-difference methods, as well as in class of multistep hybrid methods. We use the following approximation of derivatives in the construction of methods for solving of the equation (3):

$$y'(x_{n+j}) \approx \frac{1}{h}\sum_{k=-m}^{k}\gamma_i^{(j)} y_{n+i} + \frac{1}{h}\sum_{k=-l}^{k}\hat{\gamma}_i^{(j)} y_{n+i+\nu_i}, (|\nu_i| \le 1, i = -l, -l+1, ..., k; \; o = 0,1,..., k) \cdot \tag{15}$$

In this case, we can write the finite-difference equations with constant coefficients as:

$$\alpha_k y_{n-k} + \alpha_{k-1} y_{n+k-1} + ... + \alpha_1 y_{n+1} + \alpha_0 y_n = h\varphi(y'_n, y'_{n+1}, ..., y'_{n+k}) \quad (16)$$

Here, $\varphi(y'_n, y'_{n+1}, ..., y'_{n+k})$ is the limited function of its arguments. It is possible to propose the following method to finding of the numerical solution of equation (3):

$$\begin{aligned}\sum_{i=0}^{k} \alpha_i y_{n+i} &= \sum_{i=0}^{k} \alpha_i g_{n+i} + h \sum_{i=0}^{k}\sum_{j=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) + \\ &+ h \sum_{i=0}^{k}\sum_{j=0}^{k} \gamma_i^{(j)} K(x_{n+j}, x_{n+l_i}, y_{n+l_i}),\end{aligned} \quad (17)$$

where $l_i = i + \nu_i$ $(|\nu_i| < 1,\ i = 0,1,2,...,k)$.

Suppose that equation (1) has a continuous solution for construction of the methods of type (2) which determined on the segment $[x_0, X]$, and the kernel of the integral is the function $K(x, z, y)$, which continuous on totality of arguments and is defined in some closed set $G = \{x_0 \le x \le z + \varepsilon \le X + \varepsilon, |y| \le b\}$. Usually some coefficients of method (2) are chosen for $\varepsilon = 0$ in the form $\beta_i^{(j)} = 0$, $\gamma_i^{(j)} = 0$ for $j < i$. There are many methods for solving equation (1). For example, the method (17) can be applied to solving of the equation (3) when $\gamma_i^{(j)} = 0 (i, j = 0,1,...,k)$ and $\sum_{i=0}^{k}\sum_{j=0}^{k} (\gamma_i^{(j)})^2 \neq 0$ (e.g. [52]–[54]). In the first case, we obtain the Adams method, and in the second case, we obtain multistep methods such as hybrid method (e.g. [20],[33],[34],[45]).

Consider the following method, which is obtained from (15), as a particular case:

$$\begin{aligned} y_{n+2} = y_n + f_{n+2} - f_n + h(K(x_n, x_n, y_n) + K(x_{n+2}, x_n, y_n) + 4K(x_{n+1}, x_{n+1}, y_{n+1}) + \\ + 4K(x_{n+2}, x_{n+1}, y_{n+1}) + 2K(x_{n+2}, x_{n+2}, y_{n+2}))/6 \end{aligned} \quad (18)$$

This method reminds us the Simpson method, but these methods have different properties. The Simpson method is obtained from the method of (7) for $k = 2$ and it is the unique method, which has accuracy $p = 4$. The method (18) also stable and has accuracy $p = 4$, obtained from the method (17) when $k = 2$ and it is not unique, which has accuracy $p = 4$.

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i g_{n+i} + h\sum_{i=0}^{k}\sum_{j=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) + h\sum_{i=0}^{k}\sum_{j=0}^{k} \gamma_i^{(j)} K(x_{n+j}, x_{n+l_i}, y_{n+l_i}),$$

where $l_i = i + \nu_i$ $(|\nu_i| < 1,\ i = 0,1,2,...,k)$.

As is known the methods, which are applied to solving equation (3), have their own advantages and shortcomings. Therefore, scientists began to construct new methods that were at the joint of one step and multistep methods and preserved their best properties in the middle of twentieth century. Such methods were called hybrid methods. These methods differ from the preceding ones by their high accuracy and extended domain of stability. Hence, investigation of hybrid methods represents both scientific and practical interest. Taking into account that advantage of hybrid methods, we consider investigation of the method (17). Note that the method (17) generalizes of many known methods.

Remark, that the coefficients of the method (17) were defined by using of the coefficients of the method (10).

Note that the accuracy of the finite-difference methods of the type (10) is determined by means of the notion of degree, since the quantity k is an order of method (10). Therefore, notion of the degree for the method (10) is determined as follows:

Definition 1. Let the function $z(x)$ is determined on the segment $[x_0, X]$ and it is sufficiently smooth. Then, the integer quantity $p > 0$ is called the degree of the method (10), if the following is held:

$$\sum_{i=0}^{k}\left(\alpha_i z(x+ih) - h\left(\beta_i z'(x+ih) + \gamma_i z'\left(x+(i+\nu_i)h\right)\right)\right) = O\left(h^{p+1}\right), \ \ h \to 0. \tag{19}$$

One of the basic problems in researching of the method (10) is the obtaining of the maximum value for the degree p. However, value of k is given in the method (10). Therefore, the specialists try to define the relation between the order and the degree of the method (10). We use the method of undetermined coefficients for defining the relation between the quantities p and k, in which uses the Taylor expansion of the functions $z(x+ih)$, $z'(x+ih)$ and $z'(x+(i+\nu_i)h)$ in the correlation (19). In this case, we receive the system (11) of algebraic equation for determination of the coefficients of the method (10).

Thus, we get a homogeneous system of nonlinear algebraic equations for determination of the values of the quantities $\alpha_i, \beta_i, \gamma_i, \nu_i$ ($i = 0,1,2,...,k$). The amount of equations equals $p+1$, but the amount of unknowns equal $4k+4$ in this system (11). As is known the system (11) has always a zero (trivial) solution. However, the trivial solution is no interest to us. Therefore, we consider the definition of non-trivial solutions of the system (11). We can prove when $p+1 < 4k+3$ the system has nontrivial solutions. Hence, it follows that $p_{\max} = 4k+2$. Usually the methods with the degree $p = 4k+2$ are unstable for $k \geq 2$. Stability of method (10) is similarly defined from the methods, obtained from the method (10) for $\nu_i = 0 \quad (i = 0,1,...,k)$.

Definition 2. The method (10) is stable if the roots of its characteristic polynomial $\rho(\lambda)$ lie inside the unit circle on the boundary of which does not have multiple roots.

As is known, the question of finding the solution of nonlinear algebraic equations is one of the most difficult processes. Therefore, we consider the case $k = 2$ and put $\alpha_2 = 1, \alpha_1 = 0, \alpha_0 = -1$ in the system (11), and by solving the obtained system of nonlinear algebraic equations, we have:

$$\beta_2 = 64/180, \ \ \beta_1 = 98/180, \ \ \beta_0 = 18/180,$$
$$\gamma_2 = 18/180, \ \ \gamma_1 = 98/180,$$
$$\gamma_0 = 64/180, \ \ l_2 = 1+\sqrt{21}/14, \ \ l_1 = 1,$$
$$l_0 = 1-\sqrt{21}/14.$$

Hence, we get the following method:

$$y_{n+2} = y_n + h(64y'_{n+2} + 98y'_{n+1} + 18y'_n)/180 + h(18y'_{n+l_2} + 98y'_{n+1} + 64y'_{n+l_0})/180. \tag{20}$$

The constructed method has the degree $p = 8$ and is stable.

Now we consider application of the method (17) for finding of the numerical solution of the equation (3). We can write this method for applying to solving equation (3) in one variant as:

$$y_{n+2} = y_n + g_{n+2} - g_n + h(64K(x_{n+2}, x_{n+2}, y_{n+2}) + 49K(x_{n+2}, x_{n+1}, y_{n+1}) + \\ + 49K(x_{n+1}, x_{n+1}, y_{n+1}) + 9K(x_{n+1}, x_n, y_n) + 9K(x_n, x_n, y_n))/180 + \\ + h(9K(x_{n+2}, x_{n+l_2}, y_{n+l_2}) + 9K(x_{n+l_2}, x_{n+l_2}, y_{n+l_2}) + 49K(x_{n+2}, x_{n+1}, y_{n+1}) + \\ + 49K(x_{n+1}, x_{n+1}, y_{n+1}) + 32K(x_{n+2}, x_{n+l_0}, y_{n+l_0}) + \\ + 32K(x_{n+1}, x_{n+l_0}, y_{n+l_0}))/180. \quad (21)$$

As it was mentioned above, the numerical solution of equation (3) may be found by the quadrature method.

Note that if we assume that $K(x,z,y) = b(z,y)$ then integral equation (3) will be equivalent to the initial value problem for the ordinary differential equations, which has the following form:

$$y' = g'(x) + b(x,y), \;\; y(x_0) = y(x_0).$$

By another way solving of the equation (3) is replaced by the following one:

$$y' = g'(x) + K(x,x,y) + \psi(x),$$
$$\varphi(x) = \int_{x_0}^{x} K'_x(x,s,y(s))ds.$$

We can find the value of the function $\psi(x)$ by using one of the methods applied to determination of solution of the integral equations, and then taking into account the obtained values in the differential equation, we can find approximate values of solution of the integral equation (3).

Assume that solution of equation (3) is known. Then, taking this into account, we get an identity, from which we have:

$$y'(x) = g'(x) + K(x,x,y(x)) + \int_{x_0}^{x} K'_x(x,s,y(s))ds, \quad x \in [x_0, X]. \quad (22)$$

The obtained the integro-differential equation is equivalent to the following system of the differential and integro-differential equations:

$$y' = g'(x) + K(x,y,y) + v(x), \quad (23)$$

$$v' = K'_x(x,x,y) + \int_{x_0}^{x} K''_{x^2}(x,s,y(s))ds. \quad (24)$$

Remark, that solving of the integral equation is reduced to solving of the differential equations if $K''_{x^2}(x,z,y) = 0$. Several of papers are devoted to the investigation of the systems of the equations (23) and (24) (e.g. [34], [42], [45]).

Let us consider application of the method (10) to solving following example:

1. $y(x) = 1 + \frac{x^2}{2} + \int_0^x y(s)ds$. Exact solution: $y(x) = 2e^x - x - 1$. (1')

2. $y(x) = e^{-x} + \int_0^x e^{-(x-s)} y^2(s)ds$. Exact solution: $y(x) = 1$. (2')

The results of calculations are accommodated in Table 2.

Table 2. Comparison of errors for stepsize $h = 0.02$

Step size	Variable x	Error for example (1')	Error for example (2')
$h = 0.02$	0.10	$0.26E-10$	$0.15E-06$
	0.20	$0.54E-10$	$0.54E-06$
	0.30	$0.83E-10$	$0.10E-05$
	0.40	$0.11E-09$	$0.17E-05$
	0.50	$0.14E-09$	$0.24E-05$
	0.60	$0.18E-09$	$0.32E-05$
	0.70	$0.21E-09$	$0.40E-05$
	0.80	$0.25E-09$	$0.48E-05$
	0.90	$0.30E-09$	$0.57E-05$
	1.00	$0.34E-09$	$0.66E-05$

2.3 Application of Finite-difference Methods to Solving of Initial Value Problems for Integro-differential Equation of Volterra Type

As is known Volterra has shown that the mathematical models for some seasonal diseases such influenza is formulated by the integral and differential equations (e.g. [29]). This work gave impetus to the development of approximate methods for solving of the integro-differential equations. Vito Volterra is a founder of the theory of integro-differential equations. He first constructed the integral-differential equation, which is applied to the solution of some applied problems. One of popular methods for solving of the Volterra integro-differential equations is the method of quadrature. Note that the quadrature method was first used by Volterra for solving of the integro-differential equations with the variable boundaries. We consider the following initial-value problem in the Volterra integro-differential equations:

$$y' = f(x, y) + \int_{x_0}^{x} K(x, s, y(s))ds, \; y(x_0) = y_0, \quad x_0 \le s \le x \le X. \tag{25}$$

where $y(x)$ is the solution of the problem by using the following notation:

$$z(x) = \int_{x_0}^{x} K(x, s, y(s))ds, \quad x_0 \le s \le x \le X, \tag{26}$$

problem (25) can be rewritten as:

$$y' = f(x, y(x)), \; y(x_0) = y_0. \tag{27}$$

The following k-step method with constant coefficients can be applied for using the known quantities $y_1, y_2, ..., y_{k-1}$ and $z_1, z_2, ..., z_{k-1}$ for solving the problem (25). In this case, we have:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i F(x_{n+i}, y_{n+i}, z_{n+i}). \tag{28}$$

Here, y_m, z_m $(m = 0,1,2,...)$ are the approximate values of the function $y(x)$ and $z(x)$ on the mesh points $x_m = x_0 + mh$, $(m = 0,1,2,...)$, where the step size $h > 0$ is the integration step size, which is divided the segment $[x_0, X]$ into N equal parts. If there is a way to determine z_{n+k} $(n \ge 0)$, then it should be used in formula (28). Furthermore, we can calculate the values of the function $y(x)$ on the mesh points x_{n+k} $(n = 0,1,2,...,N-k)$. In this case, solving of the problem (25) is equivalent to solving an initial-value problem for ordinary differential and the integral equations (see, e.g., [46-49]). Thus, we can find solution of the problem (25) by means of multistep methods with constant coefficients. Note that solving of the integral equations can be accomplished with several different approximation methods (see, e.g., [50-51]).

To solve problem (25) one can use the following multistep method (e.g., [52] or [45]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\gamma_i z_{n+i} \, , \tag{29}$$

$$\sum_{i=0}^{k}\hat{\alpha}_i z_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}). \tag{30}$$

This method is obtained by using a multistep method to solve both of the integral equation (3) and the initial-value problem (1)-(2). Therefore, solving of the problem (25) can be accomplished by one of the approximate methods of the ordinary differential equations with using some combinations of the methods which proposed for solving of the integral equations with the variable boundaries. The order of accuracy of the stable methods which has been constructed by the scheme (29) - (30), is not exceed $k+2$; this result was established by Dahlquiste (e.g. [12]). Therefore, scientists had proposed various ways to construct stable methods with an order of accuracy higher than $k+2$. To this end, the hybrid method, which was the first investigated by Gear and Butcher, was applied to solve the problem (1)-(2) (e.g. [9,10]). However, the existence of stable forward jumping methods with a higher order of accuracy than $k+2$ was proven and the method for solving of the Volterra integral equations was proposed in [43]. Remark that the stable hybrid methods with a higher order of accuracy than $2k$ were constructed in [14], but a hybrid method was applied to extend Makroglou's ideas for solving the equation (3) in the paper [34]. Thus, we find that the numerical methods of the ordinary differential equations can be applied to solve both of integral equations of type (2) and initial-value problems by the form of (1)-(2). Note that if someone wants to solve the Volterra integral equations by using quadrature or other methods which are different from method (29)-(30), then he can't use the methods of the ordinary differential equations, exclusively to solve problem (25). However, if the kernel of the integral is the degenerate function, i.e., if

$$K(x,z,y) = \sum_{\nu=1}^{m} a_\nu(x) b_\nu(z,y) \tag{31}$$

then problem (25) can be reduced to a system of the ordinary differential equations. In this case, problem (25) can be solved by using the methods of ordinary differential equations.

In this work, we constructed stable hybrid methods with a high order of accuracy in which use minimal amount of information about solution of the problem (25).The proposed work is a continuation of the investigations conducted in [16].

Consider the application of the method (10) for solving of the problem (25). In this case, we obtained the following:

$$\begin{aligned}\sum_{i=0}^{k}\alpha_i y_{n+i} &= h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{j=0}^{k}\sum_{i=0}^{k}(\beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}) + \\ &+ \gamma_i^{(j)} k(x_{n+j+\nu_i}, x_{n+i+\nu_i}, y_{n+i+\nu_i}))(|\nu_i| < 1; i = 0,1,...,k).\end{aligned} \tag{32}$$

from which we can obtain many well-known hybrid methods. Method (10) is applied for solving of the problem (27) in [54], and it has proved that there exist stable methods of type (10) with degree $p = 3k+1$.

We can prow that the following relation between the coefficients $\beta_i^{(j)}, \gamma_i^{(j)}$ $(i, j = 0,1,...,k)$ and the quantities β_i, γ_i $(i = 0,1,...,k)$ exist:

$$\sum_{j=0}^{k}\beta_i^{(j)}=\hat{\beta}_i;\ \ \sum_{j=0}^{k}\gamma_i^{(j)}=\hat{\gamma}_i\quad (i=0,1,...,k). \tag{33}$$

Let us consider the application of the following method:

$$y_{n+1}=y_n+h(y'_{n+1/2+\alpha}+y'_{n+1/2-\alpha})/2,\ \ (\alpha=\sqrt{3}/6). \tag{34}$$

If we applied the method (10) for solving of the problem (25), then in one variant obtained the following:

$$\begin{aligned}y_{n+1}=y_n+h(f_{n+1/2+\alpha}+f_{n+1/2-\alpha})+\frac{h}{4}(K(x_{n+1},x_{n+1/2+\alpha},y_{n+1/2+\alpha})+\\+K(x_{n+1/2+\alpha},x_{n+1/2+\alpha},y_{n+1/2+\alpha})+K(x_{n+1},x_{n+1/2-\alpha},y_{n+1/2-\alpha})+\\+K(x_{n+1/2-\alpha},x_{n+1/2-\alpha},y_{n+1/2-\alpha}))\end{aligned} \tag{35}$$

Let us consider the applying of the hybrid method to solving of the next problem:

$$y'=(4\exp(-y)-x^3)/3+\frac{4}{3}\int_1^x\frac{1}{x}s^2\exp(y(s))ds,1\le x\le 2,y(1)=0$$

(the exact solution is $y(x)=\ln x$).

This problem is solved by the classical Simpson methods and by the following:

$$y_{n+1}=y_n+h(y'_n+4y'_{n+1/2}+y'_{n+1})/6 \tag{36}$$

Note that the received results consistent with the theoretical results and presented here, which can be found in Table 3.

Table 3. Comparison of errors for stepsize $h=1/32$

Step size	**Variable x**	**Error of the Simpson method**	**Error of the hybrid method**	**Error of the method (36)**
$h=1/32$	1.031	0.85E-07	0.63E-09	0.93E-03
	1.250	0.23E-05	0.10E-06	0.57E-02
	1.500	0.60E-05	0.27E-06	0.13E-01
	1.750	0.77E-05	0.32E-06	0.25E-01
	2.000	0.29E-05	0.25E-07	0.45E-01

3. CONCLUSION

In this paper, we used the method of finite-difference for solution of the ordinary differential equations, integral equations and integro-differential equations of Volterra type. In this way, we have defined some relationship between the above-mentioned equations. Note that these methods have been applied to determination of the relationship between the ordinary differential equations and Volterra integral equations by using of the mechanical quadrature, which is called quadrature method.

The method, which here is proposed for the solution of integral and integro-differential equations of Volterra type, almost the same as the method that is applied for the solution of the ordinary differential equations. Such method allows decreasing the amount of the computing work in the investigation of the integral and integro-differential equations of Volterra type.

Taking into account, the above-described relationship between the ordinary differential, integral and integro-differential equations for solving them, here, the hybrid methods are constructed with high accuracy.

We hope to see, constructed here hybrid methods will find their application in many of research projects.

The authors are most grateful to the readers, if they find the opportunity to send us your comments on the content and formulation of work.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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