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An Analytic Investigation of the Quadratic Nonlinear Oscillator by an Iteration Method

B. M. Ikramul Haque[1*] and Md. Rakib Hossain[1]

[1]*Department of Mathematics, Khulna University of Engineering & Technology, Khulna-9203, Bangladesh.*

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Abstract

A modified approximate analytic solution of the quadratic nonlinear oscillator "$\ddot{x}+x^2\,\mathrm{sgn}(x)=0$" has been obtained based on an iteration procedure. Here we have used the truncated Fourier series in each iterative step. The approximate frequencies obtained by our technique shows a good agreement with the exact frequency. The percentage of error between exact frequency and our fourth approximate frequency is as low as 0.00003%.

Keywords: Nonlinear oscillator; quadratic nonlinear oscillator; iteration method.

1 Introduction

The latter half of twentieth century saw remarkable advance in our understanding of physical systems governed by nonlinear equations of motion. Nonlinear problems have many important applications in several aspects of mathematical-physical sciences as well as other natural and applied sciences. Most natural systems are nonlinear and not nice. In this situation Perturbation method, Homotopy method, Homotopy perturbation method, Harmonic balance method, Iteration method, etc are used to find approximate solutions to nonlinear problems.

The perturbation method is the most widely consumed method in which the nonlinear term is small. The method of Lindstedt-Poincare (LP) [1-3], Homotopy method [4-7], Homotopy perturbation method [8] and Differential Transform method [9-11] are most important among all perturbation methods. An important aspect of various perturbation methods is their relationship with each other. Among them, Krylov and

Corresponding author: E-mail: b.m.haque75@gmail.com;

Bogoliubov are certainly to be found most active. In most treatments of nonlinear oscillations by perturbation methods only periodic oscillations are treated, transients are not considered. Krylov and Bogoliubov [2] have introduced a new perturbation method to discuss transients.

Harmonic balance method is another technique for finding the periodic solutions of a nonlinear system. If a periodic solution does not exist of an oscillator, it may be sought in the form of Fourier series and its coefficients are determined by requiring the series to satisfy the equation of motion. HB method which is originated by Mickens [12] and farther work has been done by Mickens [13-15], Lim and Wu [16], Hu [17], Hu and Tang [18], Wu et al. [4], Gottlieb [8], Hosen [19] and so on for solving the strong nonlinear problems. However, in order to avoid solving an infinite system of algebraic equations, it is better to approximate the solution by a suitable finite sum of trigonometric function. This is the main task of harmonic balance method. Thus approximate solutions of an oscillator are obtained by harmonic balance method using a suitable truncation Fourier series. The method is capable to determining analytic approximate solution to the nonlinear oscillator valid even for the case where the nonlinear terms are not small i.e., no particular parameter need exist.

Recently, some authors consume an iteration procedure [20-24] which is valid for small together with large amplitude of oscillation, to attain the approximate frequency and the harmonious periodic solution of such nonlinear problems. Beside this method, there are some methods [25-27] which are used to find approximate solution in the case of large amplitude of oscillations.

Iterative technique is also used as a technique for calculating approximate periodic solutions and corresponding frequencies of truly nonlinear oscillators for small and as well as large amplitude of oscillation. The method was originated by R.E. Mickens in 1987. In the article Xu and Cang [28] provided a general basis for iteration method as they are currently used to calculate the approximate periodic solutions of various nonlinear oscillatory successfully. Further Mickens used the iterative technique to calculate a higher-order approximation to the periodic solutions of a conservative oscillator.

Here the iteration technique for the determining the approximate solution of quadratic nonlinear oscillator will be presented. The obtained result will compared with the existing various results.

2 Methodology

Let us suppose that the nonlinear oscillator

$$\ddot{x} + f\left(\ddot{x}, x\right) = 0\,,\ x(0) = A,\ \dot{x}(0) = 0\,, \tag{1}$$

where over dots denote differentiation with respect to time, t.

We choose the natural frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of Eq. (1), we obtain

$$\ddot{x} + \Omega^2 \mathrm{x} = \Omega^2 \mathrm{x} - f(\ddot{x}, x) \equiv G(x, \ddot{x})\,. \tag{2}$$

Now, we formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k);\ k = 0,1,2,3,............, \tag{3}$$

together with initial condition

$$x_0(t) = A\cos(\Omega_0 t)\,. \tag{4}$$

Hence x_{k+1} satisfies the initial conditions

$$x_{k+1}(0)=A, \quad \dot{x}_{k+1}(0)=0. \tag{5}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$.

The above procedure gives the sequence of solutions: $x_0(t),\ x_1(t),\ x_2(t),\cdots$.

The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order.

At this point, the following observations should be noted:

(a) The solution for $x_{k+1}(t)$ depends on having the solutions for k less than $(k+1)$.
(b) The linear differential equation for $x_{k+1}(t)$ allows the determination of Ω_k by the requirement that secular terms be absent. Therefore, the angular frequency, "Ω" appearing on the right-hand side of Eq. (3) in the function $x_k(t)$, is Ω_k.

2.1 Solution Procedure

Let us consider the quadratic nonlinear oscillator

$$\ddot{x}+x^2\,\mathrm{sgn}(x)=0, \tag{6}$$

where

$$\mathrm{sgn}(x)=\begin{cases}1, & x>0\\ -1, & x<0\end{cases}\cdot \tag{7}$$

We choose the case $x>0$, therefore (6) becomes

$$\ddot{x}+x^2=0. \tag{8}$$

Now the iteration scheme is according to Eq. (3)

$$\ddot{x}_{k+1}+\Omega_k^2 x_{k+1}=\Omega_k^2 x_k - x_k^2. \tag{9}$$

The initial condition is rewritten as

$$x_0(t)=A\,cos\theta, \tag{10}$$

where $\theta=\Omega_0 t$. For $k=0$, the Eq. (9) becomes

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A\,cos\theta - A^2 cos^2\theta. \tag{11}$$

Now expanding $cos^2\theta$ in a Fourier Cosine series in the interval $[0,\pi]$, Eq. (11) reduces to

$$\ddot{x}_1+\Omega_0^2 x_1=\Omega_0^2 A cos\theta - A^2(0.848826 cos\theta+0.169765 cos3\theta-0.0242522 cos5\theta \\ +0.00808406 cos7\theta-0.00367457 cos9\theta+0.00197862 cos11\theta) \tag{12}$$

To check secular terms in the solution, we have to remove $cos\theta$ from the right hand side of Eq. (12).

Thus we have

$$\Omega_0=0.921318\sqrt{A}\,. \tag{13}$$

Then solving Eq. (12) and satisfying the initial condition $x_1(0)=A$, we obtain

$$x_1(t)=A(0.976027 cos\theta+0.025 cos3\theta-0.00119048 cos5\theta \\ +0.000198413 cos7\theta-0.0000541126 cos9\theta+0.000019425 cos11\theta) \tag{14}$$

This is the first approximate solution of Eq. (8) and the related Ω_1 is to be determined.

The value of Ω_1 will be obtained from the solution of

$$\ddot{x}_2+\Omega_1^2 x_2=\Omega_1^2\, x_1-x_1^2\,. \tag{15}$$

Substituting $x_1(t)$ from Eq. (14) into the right hand side of Eq. (15), we obtain

$$\ddot{x}_2+\Omega_1^2 x_2=\Omega_1^2 A(0.976027\cos\theta+0.025\cos 3\theta-0.00119048\cos 5\theta \\ +0.000198413\cos 7\theta-0.0000541126\cos 9\theta+0.000019425\cos 11\theta) \\ -A^2(0.817358\; cos\theta+0.192994 cos3\theta-0.0143718 cos5\theta \\ +0.00581554\cos 7\theta-0.00276284\cos 9\theta+0.00152151\cos 11\theta)\;. \tag{16}$$

Again absence of secular terms requires

$$\Omega_1=0.9151114\sqrt{A}\,. \tag{17}$$

Then solving Eq. (16) and satisfying initial condition, we obtain the second approximate solution,

$$x_2(t)=A(0.974868\cos\theta+0.0256823\cos 3\theta-0.000665468\cos 5\theta \\ +0.000140543\cos 7\theta-0.0000405632\cos 9\theta+0.0000149787\cos 11\theta\cdot \tag{18}$$

In a similar way we have the third approximate solution be

$$x_3(t)=A(0.9748\;\cos\theta+0.0257217\;\cos 3\theta-0.000640443\;\cos 5\theta \\ +0.000144427\;\cos 7\theta-0.0000410693\;\cos 9\theta+0.0000150912\;\cos 11\theta)\cdot \tag{19}$$

Whereas the frequencies Ω_2 and Ω_3 are

$$\Omega_2 = 0.914705\sqrt{A}\,, \tag{20}$$

$$\Omega_3 = 0.914681\sqrt{A}\,, \tag{21}$$

Thus $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ can be obtained by Eqs. (13), (17), (20), and (21) respectively, which represent the approximation of frequencies of oscillator (8).

3 Results and Discussion

An Iteration method is developed based on Mickens [20] to solve 'quadratic nonlinear oscillator'. In this section, we express the accuracy of the modified technique of iteration method by comparing with the existing results from different methods and with the exact frequency of the oscillator. To show the accuracy, we have calculated the percentage errors (denoted by Er (%)) by the definitions.

$$Er = \left| 100\{\Omega_e(A) - \Omega_i(A)\} / \Omega_e(A) \right| \;\; ; i = 0, 1, 2, 3, \ldots..,$$

where Ω_i represents the approximate frequencies obtained by the adopted method and Ω_e represents the corresponding exact frequency of the oscillator.

Herein we have calculated the first, second, third and fourth approximate frequencies which are denoted by $\Omega_0, \Omega_1, \Omega_2$ and Ω_3 respectively. A comparison among the existing results showed by Mickens and Ramadhani [30], Belendez et al. [29] and Hosen [19] with our obtained results in the following table.

It is noted that Mickens and Ramadhani [30] found only second approximate frequencies by harmonic balance method. Belendez et al. [29] found up to third approximate frequencies by using modified He's homotopy perturbation method again Hosen [19] found up to third approximate frequencies by using modified harmonic balance method.

Table 1. Comparison of the approximate frequencies obtained by the present technique and other existing results with exact frequency Ω_e (Belendez et al. [29]) of quadratic nonlinear oscillator

Exact frequency Ω_e	$0.914681\sqrt{A}$			
Amplitude A	**First approximate frequency Ω_0 Er (%)**	**Second approximate frequency Ω_1 Er (%)**	**Third approximate frequency Ω_2 Er (%)**	**Fourth approximate frequency Ω_3 Er (%)**
Mickens & Ramadhani [30]	$0.921318\sqrt{A}$ 0.73	$0.914044\sqrt{A}$ 0.070	–	–
Belendez et al. [29]	$0.921318\sqrt{A}$ 0.73	$0.914274\sqrt{A}$ 0.045	$0.914711\sqrt{A}$ 0.0032	–
Hosen [19]	$0.921318\sqrt{A}$ 0.73	$0.914427\sqrt{A}$ 0.028	$0.914733\sqrt{A}$ 0.0056	–
Adopted method	$0.921318\sqrt{A}$ 0.73	$0.915114\sqrt{A}$ 0.047	$0.914705\sqrt{A}$ 0.0026	$0.914681\sqrt{A}$ 0.00003

In our study in the above table, it is seen that the forth-order approximate frequency obtained by adopted method is almost same with exact frequency. It is found that, in most of the cases our solution gives significantly better result than other existing results. The advantages of this method include its simplicity and computational efficiency.

3.1 Convergence and Consistency Analysis

We know the basic idea of iteration methods is to construct a sequence of solutions x_k (as well as frequencies Ω_k) that have the property of convergence

$$x_e = \lim_{k \to \infty} x_k \quad \text{Or,} \quad \Omega_e = \lim_{k \to \infty} \Omega_k$$

Here x_e is the exact solution of the given nonlinear oscillator.

In the present method, it has been shown that the solution yield the less error in each iterative step compared to the previous iterative step and finally $\left|\Omega_3 - \Omega_e\right| = \left|0.914681 - 0.914681\right| < \varepsilon$, where ε is a small positive number and A is chosen to be unity. From this, it is clear that the adopted method is convergent.

An iterative method of the form represented by Eq. (3) with initial guesses given in Eq. (4) and Eq. (5) is said to be consistent if $\lim_{k \to \infty} \left|x_k - x_e\right| = 0$ Or, $\lim_{k \to \infty} \left|\Omega_k - \Omega_e\right| = 0$.

In the present analysis we see that

$$\lim_{k \to \infty} \left|\Omega_k - \Omega_e\right| = 0 \text{ as } \left|\Omega_3 - \Omega_e\right| = 0 .$$

Thus the consistency of the method is achieved.

4 Conclusion

The iteration method is a powerful and effective mathematical tool in solving nonlinear differential equation in mathematical physics, applied mathematics and engineering. In this article, the iteration method has been employed for analytic treatment of the quadratic nonlinear differential equation. The performance of this method is reliable, simple and gives many new solutions. The results obtained by the technique presented here are not only fit to be used in the case of small nonlinearities but also fit to be used in the case of high nonlinearities.

Competing Interests

Authors have declared that no competing interests exist.

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