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New Conditions That Guarantee Uniform Asymptotically Stable and Absolute Stability of Singularly Perturbed Systems of Certain Class of Nonlinear Differential Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Author EP designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author AO managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

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Abstract

The objectives of this paper is to investigate singularly perturbed system of the fourth order differential equations of the type, $\frac{dx}{dt} = f(t, x, y, \mu)$, $\mu\frac{dy}{dt} = g(t, x, y, \mu)$ to establish the necessary and sufficient new conditions that guarantee, uniform asymptotically stable, and absolute stability of the system. The Liapunov's functions were the mathematical model used to establish the main results of this study. The study was motivated by some authors in the literature, Grujic LJ.T, and Hoppensteadt, F., and the results obtained in this study improves upon their results to the case where more than two arguments was established.

Keywords: Uniform asymptotical stable; absolute stability; singularly perturbed; differential equations.

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1 Introduction

Singular perturbation technique, at the third international congress of mathematician in Heidelbery in 1904 by L. Prandtl through his paper on fluid motions with small friction, was made known to the world. Perturbation technique first appeared in Celestial Mechanics while deducing the planetary motions if only the sun and one planet are considered; the result is an elliptical motion with sun at the focus. The perturbation problem mentioned above is categorized into two cases; Regular and Singular perturbation problem. This study focuses on singular perturbation;

Singular perturbation problem is said to occur whenever the regular perturbation limit

$y\varepsilon(x) \to y\,0(x)$ fails. For example, consider the problem $\varepsilon\,\frac{dx}{dt} + x = 1$ whose solution is given by $x(t,\varepsilon) = 1 + [x(0) - 1)\exp\,(\frac{-t}{\epsilon})$ where $x(0)$ stands for initial value of x at $t = 0$. For $\epsilon > 0$ the solution tends to 1 for any $t > 0\ as\ \varepsilon \to 0$. As $\varepsilon \to 0, x(t,\varepsilon)$ increases monotonically towards the constant limit 1 for each $t > 0\ if\ \ x(0) < 1$. For $t = 0\ x(t,\varepsilon) \to x(0); t > 0, x(t,\varepsilon) \to 1$

Thus $x(t,\varepsilon)$ has a discontinuous limit as $\varepsilon \to 0$. This type of singularity, generally, occur when ε is multiplied with the highest derivative in the system. Singularly- perturbed systems are known to be rather widely used in the engineering and technology as models of real processes. Introductory Mathematical background for perturbation technique is excrescency covered in references [1,2,3,4,5]. For application in engineering and technology see e.g survey by [6,7,8], and Stability properties of SPS see [9,10,11,12,13,14,15,7,8,16,17,18].

We investigate singularly perturbed systems of order fourth order which has been reduced to order two, written in the form of nonlinear differential equations of the type (1.1) (1.2) with two arguments to establish new conditions that guarantee uniform asymptotical stable and absolute stability of the two arguments $(x^T, y^T)^T = 0$ of the systems (1.1), (1.2), with the application of Liapunov's direct method.

Consider the nonlinear differential equations of the type;

$$\frac{dx}{dt} = f(t,x,y,\mu), \tag{1.1}$$

$$\mu\frac{dy}{dt} = g(t,x,y,\mu), \tag{1.2}$$

Where,

$(x^T,\ y^T)^{\mathrm{T}}$ is a vector of state of the whole system, $x\ \in\ R^n, y \in\ R^m$,
$f \in \mathrm{C}\,(R\ \ \mathrm{X}\, R^n\ \mathrm{X}\ R^m\ \mathrm{X}\, \mathcal{M}, R^n), g \in \mathrm{C}(R\ \mathrm{X}\ R^n\ \mathrm{X}\, R^m,\ \mathrm{X}\, \mathcal{M}, R^m)$. The parameter μ is positive and is supposed to be arbitrarily small. We set $\mu \in (0,1] = \mathcal{M}$

The states $x = 0\ and\ y = 0$ have open connected neighborhoods $\mathcal{N}_x \subseteq R^n\ and\ \mathcal{N}_y \subseteq R^m$ respectively. The vector function $f\ and\ g$ are such that for $(x^T,\ y^T)^{\mathrm{T}} =\ 0$, system (1.1), (1.2) has the only equilibrium state in the Cartesian product $\mathcal{N}_x\,\mathrm{X}\ \mathcal{N}_{\mathrm{y}}$ of the sets $\mathcal{N}_x\ and\,\mathcal{N}_{\mathrm{y}}$ for any $\mu \in (0,1]$. If μ takes zero value, the system (1.1), (1.2) degenerates into the system which is described by the differential and algebraic equation.

$$\frac{dx}{dt} = f(t,x,y,0), \tag{1.3}$$

$$0 = g(t,x,y,0). \tag{1.4}$$

It is supposed that $g(t,x,y,0)$ vanishes for any $t\ \in R$ and $x \in\ \mathcal{N}_x$, Iff $y = 0$.

This study was motivated by Hoppensteadt, and Lur'e-postnikov, see [16,12], and results obtained in this study improve upon their results. They considered one argument in the case when $x = 0$ and applied Liapanov's function, their results is important in the stability investigation of systems (1.1), (1.2). This requirement is motivated by the application of the Liapanov's coordinate's transformation by Hoppensteadt [12], in the investigation of Singularly Perturbed systems. The system of lower order

$$\frac{dx}{dt} = f(t, x, 0,0). \tag{1.5}$$

Obtained in his result is important in the Stability investigation of the system of the form (1.1), and (1.2). If $\mu > 0$ is a sufficiently small value of the parameter, the system (1.1), (1.2) consists of the parts which accomplish slow and quick motions. The quick system S_τ (or the boundary layer) is obtained from (1.1), (1.2) after the change of the time scale by introducing the variable

$$\tau = (t - t_0)\mu^{-1}.$$

Then, the quick system corresponding to the system (1.2) becomes;

$$\frac{dy}{d\tau} = g(\alpha, b, y, 0). \tag{1.6}$$

In this system $\alpha \; and \; b$, $b = (\beta_{1\,,} - -\beta_n)$, are scalar and vector parameters introduce instead of $t \in R \; and \; x \in \mathcal{N}_x$ respectively. We suppose as earlier, that vanishes for any $t \in R, x \in \mathcal{N}_x$, $\mu \in (0,1]$ if and only if y = 0. The separation of the time – scales in the investigation of stability of system (1.1), (1.2) is essential due to the fact that the analysis of the degenerate system S_0 (1.5) and the quick system S_τ (1.6), are simpler problems in comparison with the general problem of stability of system (1.1), (1.2).

2 Preliminary Notes

2.1 Asymptotic stability conditions

Let $\mathcal{N}_{x0} = \{x\colon x \in \mathcal{N}_x , x \neq 0\}$, $\mathcal{N}_{y0} = \{y\colon y \in \mathcal{N}_y \; y \neq 0\}$.

The function $v(\alpha, b, y) \in C^{(1,1,1)}\,(R \text{ X } R^n \text{ X } R^m, R)$ and

$$v_\alpha = \frac{\partial v}{\partial \alpha}, \quad v_b = \left(\frac{\partial v}{\partial \beta 1}, \frac{\partial v}{\partial \beta 2}, - -, \frac{\partial v}{\partial \beta n}\right)^{\mathrm{T}}.$$

We introduce two assumptions on systems (1.5) and (1.6) connected with positive definite functions $\theta \; and \; v$.

Assumption 2.1. If there exist the following conditions:

(1) A decreasing positive definite on $\mathcal{N}_x$, and radically unbounded for $\mathcal{N}_x = R^n$ function $\theta \in C^{(1,1)}\;(R \times \mathcal{N}_{x0}, R_+)$;

(2).Positive definite function $\varphi \in C(R^n, R_+)$ and $\psi \in C(R^n, R_+) \; on \; \mathcal{N}_x \; and \; \mathcal{N}_y$, respectively.

(3) Non – negative numbers $\xi_1 \; and \; \xi_2$, $\xi_1 < 1$, and the conditions are satisfied:

(a) $\theta_t(t,x) + \theta_x{}^{\mathrm{T}}(t,x)\, f(t,x,0) \leq -\varphi(x) for \; all \; (t,x) \in R \times \mathcal{N}_{x0}$;
(b) $\theta_x{}^{\mathrm{T}}\,(t,x)[\,f(t,x,y,\mu) - f(t,x,y,0)] \leq \xi_1\, \varphi(x) + \xi_2\, \psi(y), for \; all \; (t,x,y,\mu) \in R \times \mathcal{N}_{x0} \times \mathcal{N}_{y0} \times \mathcal{M}$.

Conditions (1) - (3) (a) of Assumption 2.1 ensure uniform asymptotic stability of $x = 0$ of system (1.5) in the whole, when $\mathcal{N}_x = R^n$. Conditions (3) (b) is a requirement for the qualitative properties of the vector – function f on $\mathcal{N}_x \times \mathcal{N}_y$.

Assumption 2.2. If there exist the following conditions:

(1) A decreasing positive define on $\mathcal{N}_x \times \mathcal{N}_y$ and radically unbounded in y uniformly relatively $x \in \mathcal{N}_x$ for $\mathcal{N}_y = R^m$ function $v(t,x,y) \in C^{(1,1)} \quad (R \times \mathcal{N}_x \times \mathcal{N}_y, R_+)$
(Or $v(t,y) \in C^{(1,1)} \quad (R \times \mathcal{N}_{y0}, R_+)$ decreasing and positive definite on $\mathcal{N}_y$ and radically unbounded for $\mathcal{N}_y = R^m$;

(2) Non – negative numbers $\xi_{1,} \xi_2, \xi_{3,} \xi_4 \quad (\xi_1 < 1,\ \xi_2 < 1)$ and an integer $\pi > 1$;

(3) Positive definite functions $\varphi \in C(R^n, R_+)$ and $\psi \in C(R^m, R_+)$ $on\ \mathcal{N}_x\ and\ \mathcal{N}_y$ respectively and the following conditions are satisfied

(a). $v_y{}^T\ g(\alpha,b,y,0) \leq -\psi(y) for\ all (\alpha,b,y) \in R \times \mathcal{N}_x \times \mathcal{N}_y\ or\ (for\ all\ (\alpha,b,y) \in R \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{N}_{y0})$ respectively;

(b) $v_y{}^T [g(\alpha,b,y,\mu) - g(\alpha,b,y,0)] \leq c\ \xi_{1,} \mu^T \varphi(b) + \xi_2\ \varphi(y) \not\forall (\alpha,b,y,\mu) \in R \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$ or $(\not\forall (\alpha,b,y,\mu) \in R \times \mathcal{N}_x \times \mathcal{N}_{y0} \times \mathcal{M})$ respectively;

(c) $v_\alpha + v_b^T f(\alpha,b,y,\mu) \leq \xi_{3,} \varphi(b) + \xi_4\ \psi(y) \not\forall (\alpha,b,y,\mu) \in R \times \mathcal{N}_x \times \mathcal{N}_y \times \mathcal{M}$
or $(\not\forall (\alpha,b,y,\mu) \in R \times \mathcal{N}_x \times \mathcal{N}_{y0} \times \mathcal{M})$ respectively.

The constants $\xi_{1,} \xi_2, \xi_{3,}\ and\ \xi_4,$ mentioned in Assumption 2.1, 2.2 must be taken as small as possible. If the function V does not depend on x, then it is said to be positive definite on $\mathcal{N}_y$ only. If, in addition $\mathcal{N}_y$ is time invariant, the condition (c) in Assumption2.2 is omitted.

Let the equations, (1.1), and (1.2) be the Lur'e - Postnikov form of system. See, [8].

$$\frac{dx}{dt} = A_{11}x + A_{12}y + q_{1\Phi_1}(\sigma_1), \tag{2.1}$$

$$\sigma_1 = C_{11}^T x + C_{12}^T y;$$

$$\mu \frac{dy}{dx} = A_{21}x + A_{22}y + q_{1\Phi_2}(\sigma_2)$$

$$\sigma_2 = C_{21}^T x + C_{22}^T y. \tag{2.2}$$

The matrices $A(.)$ and vectors C(.) and q(.) are of the appropriate dimensions. The nonlinearityΦ_i, $i = 1,2$ are continuous, $\Phi_i(0) = 0$, and in Lur'e sector $[0, ki]$, $ki \in (0, +\infty)$ satistfy the conditions.

$$\frac{\Phi_{1\,1_i}(\sigma i)}{\sigma 1} \in [0, ki], i = 1,2; \text{for all } \sigma i \in (-\infty, +\infty)$$

The nonlinearities Φ_i are considered incidentally, for which the state $x = 0, y = 0$ is the only equilibrium state of the degenerate system

$$\frac{dx}{dt} = A_{11}x + q_1\varphi_1(\sigma_1^\circ), \sigma_1^\circ = C_{11}^T x. \tag{2.3}$$

And the system, describing the boundary layer respectively

$$\frac{dy}{dt} = A_{22y} + q_2\Phi_2(\sigma_2^\circ), \quad \sigma_2^\circ = C_{22y}^T .$$

This assumption is valid if

$$C_{ii}^T\, A_{ii}^{-1}\; q_i > 0, i = 1{,}2$$

We suppose the matrix A_{11} is stable, the pair $(A_{11,}q_1)$ is controlled and there exist numbers $\Psi_1 \in [0, +\infty]$ and $\varepsilon_1 \in (0, +\infty)$ such that

$$k_1^{-1} + R_\rho(1 + j\Psi_1\omega)C_{11}^T(A_{11} - j\omega I_1)^{-1}q_1 - \varepsilon_1 q_1^T(A_{11}^T + j\omega I_n)^{-1}(A_{11} - j\omega I_n)^{-1}q_1 \geq 0 \quad \text{for all } \omega \in [0, +\infty].$$

Then,

$$\theta(x) = (x^T H_1 x + \Psi_1 \int_0^{\sigma_1^\circ} \Phi_1\,(\sigma_1^\circ) d\sigma_1^\circ)^{\frac{1}{2}}$$

is the Liapanov's function for degenerate system (2.3) for any Φ_i taking the values in $[0, K_1]$, where H_1 is a solution of the equations.

$$A_{11}^T H_1 + H_1 A_{11} + q_1 q_1^T = -\varepsilon_1 I_1, h_1 + H_1 q_1 = -\sqrt{\gamma q_1} \tag{2.4}$$

For,

$$\gamma = K_1{}^{-1} - \xi_{1,}c_{11}^T q_1, h_1 = \frac{1}{2}(\Psi_1 A_{11}^T C_{11} + C_{11}). \tag{2.5}$$

Now we shall verify the conditions of Assumptions 2.1 and 2 .2.

The verification of conditions of Assumption 2.1:

Let H_1 and $\theta(x)$ be defined as above. Hence, the function $\theta(x)$ is decreasing positive definite on R^n and radically unbounded. We shall check up the condition (3) (a) first

(a) in the case $\theta_t = 0$ and

$\theta_x^T(x)\; f(x, 0{,}0) \leq -\frac{1}{2}\varepsilon_1\eta_2{}^{-1} \parallel x \parallel \;\; \forall(x \neq 0) \in R^n,$

Where,

$\eta_2 = A^{\frac{1}{2}}(H_1 + \frac{1}{2}\,(\Psi_1 K_1 C_{11} C_{11}^T)$ and $A(\circ)$ is a maximal Eigen value of matrix$(\circ)$. Hence
$\varphi(x) = \eta_3 \;\parallel x \parallel\;,\; \eta_3 = \frac{1}{2}\varepsilon_1\eta_2{}^{-1}$.

And

$\theta_t + \theta_x^T f(x, 0{,}0) \leq -\varphi(x)\forall(x \neq 0) \in R^n$, and besides,
$N_x = R^n, N_{x0} = \{x: x \neq 0, x \in R^n\}$;

(b) For the function $\theta(x)$ we have:

$\theta_x^T[f(x,y,\mu) - \ f(x,0,0)] = \frac{1}{2}Q(x)x^T\left(2H_1 + \xi_{1,}\frac{\Phi_1(\sigma_1^\circ)}{\sigma_1^\circ}C_{11}C_{11}^T\right) \times \quad \{A_{12y} + q_1[\Phi_1(\sigma) - \Phi_1(\sigma_1^\circ)]\} \leq \xi_{1,}\varphi(x) + \xi_{2,}(_{(\Psi_1}(y), \forall x \in \mathcal{N}_{x0}, \forall y \in R^m \ \forall \mu \in (0,1).$

Incidentally,

$\psi(y) = \rho_3\|y\|,\ \ \xi_{1,} = k_1(\eta_1\eta_3)^{-1}\ \eta_2\|q_1\|\ \|C_{11}\|$ And
$\xi_{2,} = (\eta_1\rho_3)^{-1}\ \eta_2[k_1\|C_{12}\|\ \|q_1\| + \|A_{12}\|)\ \lambda_1 = \lambda^{\frac{1}{2}}(H_1)$, where $\lambda(\circ)$ is a minimal eigen value of matrix $(\circ)$. The value$\rho_3\ > 0$ will be defined below.

The numbers ξ_1 and ξ_2 and the function $\theta,\ \varphi$ and ψ satisfy the conditions of Assumptions 2.1.

The verification of the conditions of Assumption 2.2:

We take the function $\mathrm{V}(y) = \|y\|$ as the auxiliary function. This choice shows the alternative to the choice of the liapanov functions. The function V is decreasing positive definite in R^n and radically unbounded. In order to verify condition (3)(a) of Assumption 2.2, we present the system of the boundary layer in the form suggested by Rosenbrok

$$\frac{dy}{d\tau} = D_{22}(\propto_2)y.$$

Where,

$$D_{22}(\propto_2) = A_{22} + \propto_2 (\sigma_2^\circ)q_2C_{22}^T,\ \propto_2 (\sigma_2^\circ) = \frac{\Phi(\sigma_2^\circ)}{\sigma_2^\circ}.$$

The matrix $\widehat{D}_2(\propto_2) = D_{22}^T(\propto_2) + D_{22}(\propto_2)$ is negative definite for each $(\sigma,\varphi_2) \in R\ x\ \mathcal{N}_0\ ([0,k_2])$ if and only if $D_{22}(0)$ and $D_{22}(k)$ are negative definite. Assumption 2.2 is fulfilled.

At least $\psi(y) = \rho_3(y)$ and $V_y^Tg(\propto,b,y,0) \leq -\psi(y)\ \forall(y \neq 0) \in R^m$ ensure the satisfaction of condition (3) (a) for condition (3) (a) we have

$$V_y^T[g(\propto,b,y,,\mu) - g(\propto,b,y,0)\ ] = \frac{1}{V}y^T\{\mu A_{21}b + q_2[\Phi_2(\sigma_2) - \Phi_2(\sigma_2^\circ)]\ \ \forall(y \neq 0) \in R^m.$$

If we let,

$\xi_1 = 2\varepsilon_1\eta_2$ Suppose$\|A_{21} + \propto q_2C_{21}^T\|$,
$\xi_2 = K_2\|q_2C_{21}^T\|\rho_3^{-1}$.

We assume that $\xi_2 < 1$, then

$$V_y^T[g(\propto,b,y,\mu) - g(\propto,b,0,0)] \leq \xi_1\mu\varphi(b) + \xi_2\psi(y)$$
$$\forall(\propto,b,y,\mu)\ \in\ \ R \times R^n \times R^m \times (0,\infty).$$

This corresponds to condition (3) in Assumption 2.2 for $\pi = 1$. Checking up condition (3) (c) we taking into account that $V_\propto \equiv 0$ and $V_b \equiv 0$ and therefore, $\xi_3 = 0$ and $\xi_4 = 0$ the lower estimate of the upper bound of the parameter μ changes and has the form

$$\bar{\mu} = \frac{1-\xi_{2,}}{\xi_2}$$

Now the inequality $1 > \xi_1 + \xi_1$ ensure absolute stability of the state $z = (x^T,y^T)^T = 0$ of system 2.1 and 2.2

3 The Main Results

Theorem 3.1. In order that the equilibrium state, $(x^T, y^T)^T = 0$, of system (1.1), (1.2) to be uniformly asymptotically stable, it is sufficient that conditions of assumptions 2.1 and 2.2 be satisfied for every $\mu \in (0, \bar{\mu})$, and for $\mu \to 0$. As soon as the inequality

$$1 > \xi_1 + \xi_1 \bar{\mu}^{\pi-1} + \xi_3$$

holds.

If moreover $\mathcal{N}_y \times \mathcal{N}_y = R^{m+n}$, then the equilibrium state is uniformly asymptotically stable in the whole for every $\mu \in (0, \tilde{\mu})$ and for $\mu \to 0$

Proposition 3.1. The function V is strictly decreasing in $t \in [\tau_i{}^* - 1, \tau_i]$ along motions $\xi[t; t_0, z_0, \mu]$ of system (1.1),(1.2) for every $\mu \in (0, \tilde{\mu})$ and for $\mu \to 0$

Theorem 3.2. Let the motion $(x^T(t; t_0, x_, \mu),\ y^T(t; t_0, y_0, \mu))^T$ of system (1.1) and (1.2) be continuous for the equilibrium state $(x^T, y^T)^T = 0$ of system (1.1) (1.2) be uniformly asymptotically stable for every $\mu \in (0, \mu_0)$ and for $\mu \to 0$ if it is necessary that the conditions of assumption 2.1 and 2.2 be satisfied, and it is sufficient that

(1) The matrices $A_1(\mu) + A_1^T(\mu)$ and $A_2(\mu) + A_2^T(\mu)$ be conditionally positive;
(2) The matrix $C(\mu)$ be continually negative for every $\mu \in (0, \mu_0)$ and for $\mu \to 0$

If in addition $\mathcal{N}_y \times \mathcal{N}_y, = R^{m+n}$, then the equilibrium state $(x^T, y^T)^T$ is uniformly asymptotically stable in the whole for every $\mu \in (0, \mu_0)$ and for $\mu \to 0$

Proof of the main results

Proof of theorem 3.1

Let the function V be defined by the formula $V = \theta + V$ then $V(t, x, y) \in C^{1,1,1}(R \times \mathcal{N}_{x0} \times \mathcal{N}_{y_0})$ and, since the conditions of Assumptions 2.1 and 2.2 are satisfied. It is decreasing and positive on $\mathcal{N}_X \times \mathcal{N}_y$. The Euler derivative

$\frac{dv\,(t, x(t), y(t), \mu)}{dt}$ of it along the motion of system (1.1), (1.2), $z(t) = (x^T(t), y^T(t))^T \neq 0,\ t \in t0,+\infty$ means that the equilibrium state is reachable and therefore is not considered, due to system (1.1),(1.2) is

$$\frac{dv}{dt} = \theta_t + \theta_x^T f + V_t + V_x^T f + \frac{1}{\mu} V_y^T g.$$

The right-side of this expression is transformed to the form

$$\frac{dv}{dt} = \theta_t + \theta_x^T f(t, x, 0,0) + \theta_x^T [f(t, x, y, \mu) - f(t, x, 0,0)] + V_x^T f[t, x, y, 0] + \frac{1}{\mu} V_y^T g(t, x, y, 0) + \frac{1}{\mu} V_y^T [g(t, x, y, \mu) - g(t, x, y, 0)].$$

Conditions (3) (a) and condition (3) (b) of Assumption 2.1 and (3) (a) – (3) (c) of Assumption 2.2 lead to the estimate

$$\frac{dv}{dt} \leq -(1-\xi_1\mu^{\pi-1}-\xi_3)\varphi(x) - \frac{1}{\mu}[1-\xi_2-\mu(\xi_2+\xi_4)]\psi\,(y), \forall_\mu \in (0,\tilde{\mu}) \quad \mu \to 0 \quad \forall (t,x,y) \in R \times \mathcal{N}_{x0} \times \mathcal{N}_{y0}\ . \tag{3.1}$$

Let

$$\mathcal{N}_{0x} = \{z\colon\ x=0, y\ \in\ \mathcal{N}_{y0}\},\ \mathcal{N}_{0y} = \{z\colon x\ \in\ \mathcal{N}_{x0}, y=0\},\ \mathcal{N}_0 =\ \mathcal{N}_{0x} \times \mathcal{N}_{0y}.$$

It is clear that

$$\mathcal{N}_X \times \mathcal{N}_y\ =\ \mathcal{N}_{X0} \times \mathcal{N}_{y0} \times \mathcal{N}_0 \times \{z\colon z=0\}$$

Let V_M be a maximal positive number, for which the largest connected neighborhood $\mathcal{U}_{VM}(t)$ of point $z=0$ is such that,

$$V(t,x,y) \in [0,V_M),\ V(t,y) \in\ \mathcal{U}_{VM}(t)\ , V_t\ \in R.$$

Is a subset of the product $\mathcal{N} =\ \mathcal{N}_x \times \mathcal{N}_y$ for every $t \in R$. The existence of the value $V_M > 0$ is implied by the positive definiteness of function V on $\mathcal{N}$ and the time-invariance of the neighborhood of point $z=0$.

Let $\tau_i, \tau_i^*,\quad t_0 \quad \leq \tau_i < \tau_i^* \leq +\infty$ denote the times when $z(t) \in \frac{U_{VM}(t)}{\mathcal{N}_0}\ \ \forall\ t \in (\tau_i, \tau_i^*), \tau_i > t_0$ and $z(t)\ \epsilon\ \ \mathcal{N}_0\ \forall_t\ \epsilon\ [\tau_{i-1}^*, \tau_i]$. If $z(t_0) \in \frac{U_{VM}(t_0)}{\mathcal{N}_0}$, then $i=0, \tau_0 = t_0, [\tau_0, \tau^*] = [t_0, \tau^*]$ is the first interval to be considered, and the next is (τ_0^*, τ_1). If $(t)\ \epsilon\ \ \mathcal{N}_0$, $i=1,\ \tau_0^* = t_0$ and $[\tau_0^*, \tau_1]$ is the first interval to be considered, and the next is (τ_1, τ_1^*). in what follows, $i \geq 0$ is an integer.

Let

$\xi(t;t_0,\mathrm{z}_0,\mu) = [x^T(t;t_0,z_0,\mu), \eta^T(t;t_0,z_0,\mu)]^T, \xi\ (t_0;t_0,z_0,\mu) \equiv z_0,$
is a motion of system (1.1), (1.2) for the initial value z_0 and $t=t_0$ when $\mu > 0$.

This complete the proof

Proof of Proposition 3.1

The proofs are divided into three parts;

Part 1.

Let there exist a time $\hat{t}\,\epsilon\ [\tau_{i-1}^*, \tau_i]$ when $V(t,x(t),y(t)) \leq V(\hat{t}, x(\hat{t}), y(\hat{t}))$ for some $(\tau_{i-1}, \tau_{i-1}^*)$. If $\hat{t} = \tau_{i-1}^*$, then there exist $\bar{\tau}_1, \bar{\tau}_2\ \epsilon\ \ (\tau_{i-1}, \tau_{i-1}^*), \bar{\tau}_1 < \bar{\tau}_2$ such that $V\big(\tau_1, x(\tau_i), y(\tau_1)\big) \leq V\big(\tau_2, x(\tau_2), y(\tau_2)\big)$.

Due to the continuity of function V and ξ at $t\ \epsilon \curlyvee_0, \forall\ t\ \epsilon\ R$, which ensure the continuity of function f and g. Therefore, there exist $\tau_3\,\epsilon\ [\bar{\tau}_1, \bar{\tau}_2]$ when

$$\frac{dv}{dt}\Big|_{t=\tau_3} \geq 0\ .$$

However, this contradicts estimate (3.1) because of the positive definiteness of functions φ and ψ and the fact that $(1-\xi_1-\xi_1\mu^{\pi-1}-\xi_3) > 0,\ \frac{1}{\mu}[1-\xi_2-\mu(\xi_2+\xi_4)] > 0\ \forall\ \mu\ \epsilon\ (0,\hat{\mu})$.

Hence, the equality $\hat{t} = \tau_{i-1}^{*}$ is impossible and a value $\hat{t} \epsilon (\tau_{i-1}^{*}, \tau_i)$ is to be considered. Let $T_1 \leq [\tau_{i-1}^{*}, \tau_i]$ be a set of all time t such that $x(t) = 0$ is excluded $\forall\, t \,\epsilon\, [t_0, +\infty]$, then, by virtue of the continuity of the system of motion it should be $T_1 = [\tau_{i-1}^{*}, \tau_i]$ Then $\theta(t, x(t)) = \theta(t, 0)$

$\forall\, t \epsilon\, T_1$ and $V(t, x(t), y(t)) = V(t, 0, y(t))$. Moreover,

$$\frac{d}{dt}\, V(t, 0, y(t)) = \frac{d}{dt}\, V(t, 0, y(t)) \leq -\frac{1}{\mu}(1 - \xi_2 - \xi_4)\psi(y(t))$$
$$\forall\, t \,\epsilon\, T_1, \forall_{\mu}\, \epsilon\, (0, \hat{\mu}),\ \mu \to 0 \tag{3.2}$$

This contradicts the assumption that $\hat{t} \,\epsilon\, T_1$. now let $T_2 = [\tau_{i-1}^{*}, \tau_i]$. Then $(t) = 0\ \forall\, t \,\epsilon\, T_2$. Therefore $V(t, x(t), y(t)) = V(t, x(t), 0) \quad \forall\, t \,\epsilon\, T_2$,

$$\frac{d}{dt}\, V(t, x(t), 0) = \frac{d}{dt}\, V(t, 0, y(t)) \leq -(1 - \xi_1 - \xi_3)\, \varphi(x(t))\,) \quad \forall\, t \,\epsilon\, T_2\, .$$

That contradicts the assumption that $\hat{t} \epsilon\ T_2$. In general, there exist no values $\hat{t} \,\epsilon\, [\tau_{i-1}^{*}, \tau_i]$ mentioned above.

Part 2.

Inequality (3.1), (3.2), estimates of $\bar{\mu}$ and conditions $1 > \xi_1 + \xi_1 \bar{\mu}^{\pi - 1} + \xi_3,\ \xi_2 > 0$ together with the positive definiteness of functions φ and ψ proves that the function V strictly decreases on interval $[\tau_{i-1}^{*}, \tau_i],\ \tau_{i-1}^{*} \geq t_0, \forall_i \geq 1$

Part 3.

Let there exist $\hat{t} \,\epsilon\, [\tau_{i-1}^{*}, \tau_i]$ such that

$V(t, x(t), y(t)) \geq V(\hat{t}, x(\hat{t}), y(\hat{t}))$ for some $t \,\epsilon\, (\tau_i, \tau_i^{*})$.

Hence, there exist $\bar{\tau}_1, \bar{\tau}_2 \in (\tau_i, \tau_i^{*}), \bar{\tau}_1 < \bar{\tau}_2$ such that

$V(\bar{\tau}_1, x(\bar{\tau}_1), y(\bar{\tau}_1)) \leq V(\bar{\tau}_2, x(\bar{\tau}_2), y(\bar{\tau}_2))$. Due to the continuity of $V(t, x(t), y(t))$ therefore, there exist $\bar{\tau}_3 \in [\bar{\tau}_1, \bar{\tau}_2]$ is such that

$$\frac{d}{dt} V(t, x(t), y(t))|_t = \bar{\tau}_3 \geq 0 \text{ And this contradicts Condition} \tag{3.3}$$

The combination of assertions of parts 1-3 proves proposition 3.1 In views of the positive definiteness of V we establish according to the result part 1 the uniform stability of the state $x = 0$ of system (1.1), (1.2) for $\forall\, \mu \in (0, \bar{\mu})$ and for $\mu \to 0$. Further on, because of the positive definiteness of functions φ and ψ and the fact that $(1 - \xi_1 - \xi_1 \mu^{\pi-1} - \xi_3) > 0$ and $(1 - \xi_1 - \xi_1 \mu^{\pi-1}) > 0\ \forall \mu \in (0, \bar{\mu})$ as $\mu \to 0$ and due to estimate of $\bar{\mu}$, $\frac{d}{dv} V$ is proved to be smaller than a negative definite function on $\mathcal{N}_{x0} \times \mathcal{N}_{y0}$, on $\mathcal{N}_{0x}$ and on $\mathcal{N}_{0y}$. This result together with the condition of positive definiteness and decrease of function V proves uniform attraction in the whole of the state $x = 0$ of system (1.1), (1.2) and this completes the proof of the first assertion of the theorem.

In the case when $\mathcal{N}_x \times \mathcal{N}_y = R^{m+n}$, the function V will be radically unbounded and this together with the other conditions proves the second assertion of the theorem.

This theorem is applied in the absolute stability analysis of singularly perturbed Lur'e-postnikov systems. See [8].

Proof of theorem 3.2

The proof of Theorem 3.2, is similar to that of theorem 3.1, taking into account that its conditions are equivalent to the conditions of the theorem on uniform asymptotic stability of the authors in the literature, see, [19].

The theorem is proved

Example 3.1. Let

$$A_{11} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, q_1 = \begin{pmatrix} 0 \\ 10^{-1} \end{pmatrix}, C_{11} = \begin{pmatrix} -10^{-2} \\ 0 \end{pmatrix},$$
$$A_{12} = I, C_{12} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, K_1 = 2 \ .$$

And

$$A_{21} = 10^{-3} I_2, C_{21} = \begin{pmatrix} 10^{-3} \\ 0 \end{pmatrix}, K_2 = 1$$
$$A_{22} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}, q_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C_{22} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In this example we take $\psi_1 = 1\ \varepsilon_1 = 10^{-1}$ so that

$$\frac{1}{K_1} + R_e(1 + j\psi_1\omega)C_{11}^T(A_{11} - j\omega I_2)^{-1}q_1 - \varepsilon_1 q_1^T(A_{11}^T + j\omega I_2)^{-1}$$
$$(A_{11} - j\omega I_2)^{-1}q_1 \equiv \frac{1}{K_1} > 2 \ .$$

Further

$$g_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, H_1 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

is defined from the equation

$$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} + \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = -\frac{1}{10}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the form,

$$H_1 = \frac{1}{20}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hence,

$\eta_1 = 0.16$ and $\eta_2 = 0.45$ the matrix $\widehat{D}_{22}(\propto_2)$ reads

$$\widehat{D}_{22}(\propto_2) = \begin{pmatrix} -8 + 2 \propto_{22} & 2 + \propto_{22} \\ 2 & -8 \end{pmatrix}.$$

The matrices $\widehat{D}_{22}$ and $\widehat{D}_{22}(1)$ are negative definite.

And if $\xi_1 = 0.05, \xi_2 = 1.88, \xi_1 = 0.02$ and $\xi_2 = 0.002$. therefore

$\tilde{\mu} = 0.52$. Since $\xi_1 + \xi_2 = 0.53$ is smaller than 1, the state

$Z = (x^T, y^T)^T = 0$ Of the system defined in this example is absolutely stable for each

$\mu \epsilon\ (0; \tilde{\mu})$, That $\mu\epsilon(0; 0.\propto_2)$ on $\mathcal{N}_0(<), \leq [0, k], k = diag(2,1)$

The advantage of the separation of the time-scales in this example is that order of the system in question is diminished. Namely, instead of the system of the fourth order, we investigate two systems of the second order and verify the inequality $1 > \xi_1 + \xi_1$. Moreover, the lowering of the order of the systems simplifies the construction of the Liapanov's functions.

4 Conclusion

It is a known fact that Singular Perturbation technique which was made known by L. Prandtl in the 1904's has attracted wide attention since his original work. This study investigated Singular Perturbation of the form (1.1) and (1.2) and used Liapunov's direct (second) method to establish necessary and sufficient conditions that guaranteed the uniform asymptotical stable state and absolute stability of the systems (1.1) and (1.2). The results obtained in this study improve upon on the literature as in the case of authors in [16] and [12].

Competing Interests

Authors have declared that no competing interests exist.

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