Engineering and Applied Science Letters

PSR Press

Article

Analytical solution of isotropic rectangular plates resting on Winkler and Pasternak foundations using Laplace transform and variation of iteration method

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Received: 7 July 2019; Accepted: 21 October 2019; Published: 3 November 2019.

Abstract: Dynamic analysis of isotropic thin rectangular plate resting on two-parameter elastic foundations is investigated. The governing system is converted to system of nonlinear ordinary differential equation using Galerkin method of separation. The Ordinary differential equation is analyzed using hybrid method of Laplace transform and Variation of iteration Method. The accuracies of the analytical solutions obtained are verified with existing literature and confirmed in good agreement. Thereafter, the analytical solutions are used for parametric studies. From the results, it is observed that, increase in elastic foundation parameters increases the natural frequency. Increase in aspect ratios increases the natural frequency. It is expected that the present study will add value to the existing knowledge in the field of vibration.

Keywords: Analytical solution, deflection, Laplace variation of iteration method, natural frequency, Winkler and Pasternak.

1. Introduction

Research into vibration analysis of thin isotropic rectangular plate resting on elastic nonlinear foundation is vast gaining significant awareness among researchers due to its wide applications and important in the field of engineering. Geotechnics engineers need to understand the behaviour of plates when embedded in soil for their design, structural engineers requires same information for the design of the structural foundations likewise highway engineers rely on the information for the highway pavement design. In the design of elastic soil foundation, the adoption of two- parameter foundations gives better results than the use of Winkler foundation alone, which is associated with limitation of shear interaction among the spring elements. In the study of dynamic behavior of plates, Jain *et al.* [1] worked on free vibration of rectangular plate. In another work, natural frequency of rectangular plate was determined by Bhat [2] using Rayleight method. Few years later, Balkaya [3] investigated the dynamic response of rectangular plate using differential transform method (DTM). Thereafter, Gupta *et al.* [4] analyzed forced vibration of rectangular plate with varying thickness. In a further study, some other researchers [5–10] studied buckling and vibration of plates and beams.

Several authors already applied different method of solutions for analysis of thin rectangular plate. However, in numerical analysis [9–23], it is very important to carry out convergence and stability study which increases the computational time and cost otherwise the solution will diverge. Furthermore, exact method [24–26] are having the limitation of handling nonlinear problem due to the complex mathematics involved. These limitations had led to the introduction of semi-analytical methods. Ozturk and Coskun [27] used Homotopy perturbation method (HPM) in the study of plate dynamic behaviour. However, despite the effectiveness, there is setback of finding embedded parameters. In another study, Galerkin method of solution was adopted by Njoku [28] for vibration analysis of thin isotropic rectangular plate. The method suffers the limitation of extension of the series solution to provide precise result. In a later work, Pirbodaghi *et al.*[29] utilized Homotopy analysis method (HAM) for investigation of vibration analysis of beam. HAM suffers from limitation of assumption of solution for the expression. Variation of iteration method (VIM), was first proposed by He [30–36], has been applied to investigate many nonlinear partial differential equation. The approach uses

Lagrange multiplier to find the analytical solution with very fast convergence. This present study adopts the use of exact method to handle the linear part of the system governing equation and solving the rest of equation with very effective method of VIM. The advantage of this method over other hybrid method calls for its application in this research.

Despite the effectiveness of the method and high prediction of results, the author realized that, with several researches on dynamic analysis of plate, Laplace transform and VIM has not been used to determine analytical solution of thin rectangular isotropic plate resting on two-parameter foundations. Therefore, the present study is on determination of analytical solution of free vibration of thin isotropic rectangular plate resting on nonlinear foundation. The analytical solution obtained is used for investigation of the controlling parameters.

2. Problem formulation and mathematical analysis

Considering homogenous rectangular plate of uniform thickness resting on Winkler and Pasternak foundations as shown in Figure 1. The two opposite edge $y = 0$ and $y = b$ are regarded as simply supported.

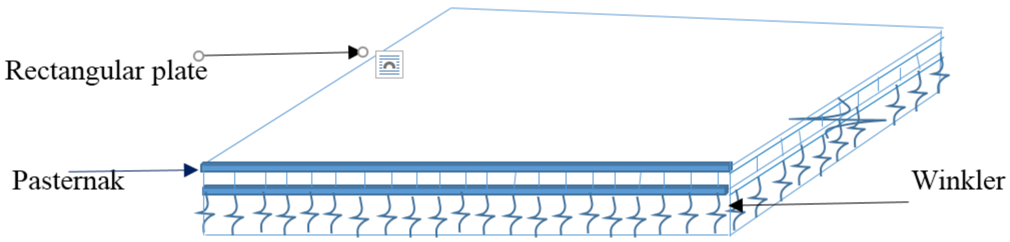

Figure 1. Rectangular plate resting on two-parameter foundations

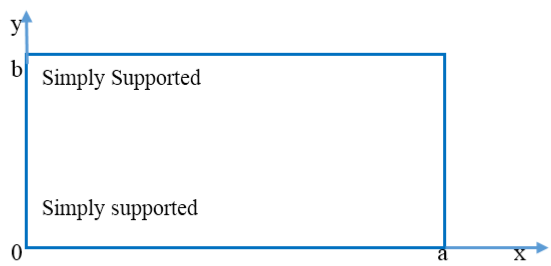

Figure 2. Geometry of plate with boundary conditions

The domain are $0 \leq x \leq a$, $0 \leq y \leq b$ where a and b represents the length and breadth of the rectangular plate as shown in Figure 2. The following assumptions are made for the development of the governing equation [37]:

1. Normal stresses in the direction transverse to the plate are considered small.
2. Thickness of plate is smaller compared to the other dimensions.
3. Plate is of constant thickness.
4. Normal to the undeformed middle surface remains straight and unstretched in length and still normal to the deformed middle surface.

The governing equation for thin isotropic rectangular plate as reported by Leissa [38] is;

$$D\left(\frac{\partial x^4 w(x,y,t)}{\partial x^4} + 2\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial x^4}\right) + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} + K_w w(x,y,t) + K_p w^3(x,y,t) = 0, \quad (1)$$

where, $w(x,y,t)$ represents the transverse deflection, D is the flexural rigidity $\frac{Eh^3}{12(1-v^2)}$, represents modulus of elasticity h, represents the plate thickness, v represents the Poisson ratio of plate material, ρ represents the mass density of the plate, ω represents the radial frequency $(rad/s), k_w$, and k_p are Winkler foundation and Pasternak foundation parameter respectively.

Using the following dimensionless variables:

$$W = \frac{w}{w_{max}}, X = \frac{x}{a}, Y = \frac{y}{b}. \tag{2}$$

According to Kantorovich type approximation, the free vibration of Equation (1) can be written as:

$$w(x,y,t) = w(x,y)e^{j\omega t}, \tag{3}$$

$$\Omega^2 = \frac{a^4 ph}{D}\omega^2, k_w = \frac{a^4 k_w}{D}, k_p = \frac{a^4 k_p w_{max}^2}{D}, \tag{4}$$

$$\frac{\partial^4 W(x,y)}{\partial X^4} + 2\lambda^2 \frac{W(x,y)}{\partial X^2 \partial Y^2} + \lambda^4 \frac{\partial^4 W(x,y)}{\partial Y^4} - \Omega^2 W(x,y) + k_w W(x,y) + k_p W^3(x,y) = 0. \tag{5}$$

Assuming the two opposite edges of Figure 1, $Y = 0$ and $Y = 1$ to be simply supported, deflection function can be represented as follows:

$$W = W(X) sin(m\pi Y). \tag{6}$$

Substituting the derivative of Equation (6) into governing differential equation

$$\frac{d^4 W(X)}{dX^4} - 2\lambda^2 m^2 \pi^2 \frac{d^2 W(x)}{dX^2} - (\Omega^2 - k_w - \lambda^4 m^4 \pi^4) W(X) + k_p W^3(X) = 0, \tag{7}$$

where $\lambda \left(\frac{a}{b}\right)$ represents the aspect ratio, m is an integer, Ω is the frequency parameter, a represents side length along $x - axis$.

2.1. Boundary conditions

Three boundary conditions are considered at $X = 0$ and $X = l$ namely, Simply supported and clamped edge (SC), Simply supported and simply supported edge (SS) and Simply supported and free edge conditions (SF).

$$Clampededge : W = \frac{dW}{dX} = 0, \tag{8}$$

$$SimplySupportededge : W = \frac{d^2 W}{dX^2} - v(\lambda^2 m^2 \pi^2) W = 0, \tag{9}$$

$$Freeedge : \frac{d^2 W}{dX^2} - v(\lambda^2 m^2 \pi^2) W = 0, \frac{d^3 W}{dX^3} - (2 - v)(\lambda^2 m^2 \pi^2) \frac{dW}{dX} = 0, \tag{10}$$

3. Method of Solution: Laplace transform and variation iteration method

3.1. Basic ideal of Laplace transform

If $f(t)$ is a function of a variable t. $\mathscr{L} F(t)$ and is defined by the integral:

$$\mathscr{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t) dt. \tag{11}$$

Some of the properties used in this study include:

$$\mathscr{L}\{1\} = \frac{1}{s}(s \geq 0), \tag{12}$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}(s \geq 0), \tag{13}$$

$$\mathscr{L}\{F^n(t)\} = n^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0)\ldots F^{n-1}(0), \tag{14}$$

where $F^n(t)$ represents the $n-th$ derivative of $F(t)$ and $\mathscr{L}\{F(t)\} = f(s)$. If Laplace transform of $F(t)$ is $f(s)$, then the inverse Laplace transform of $f(s)$ is expressed as $F(t) = \mathscr{L}^-\{f(s)\}$, where $\mathscr{L}^-$ is called inverse Laplace operator. The inverse Laplace of Equations (12) and (13) are:

$$1 = \mathscr{L}^- \frac{1}{s}, \tag{15}$$

$$t^n = \mathscr{L}^- \left(\frac{n!}{s^{n+1}}\right), \tag{16}$$

3.2. Laplace and variation iteration method

Assuming the following nonlinear differential equation:

$$Lw(x) + Nw(x) = f, \tag{17}$$

L represents the linear operator, N is nonlinear operator, f is the source or analytical function. VariatiFon iteration method use the correction function for Equation (17) as:

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\zeta)\left[Lw_n(\zeta) + N\tilde{w}_n(\zeta) - f(\zeta)\right]d\zeta, \;\; n = 0, 1, 2, \ldots, \tag{18}$$

where λ is general Lagrange multiplier identified through variational theory. The subscript n represents the nth term and $\tilde{w}_n$ is a constrained variation $(\delta\tilde{w}_n = 0)$.

Laplace transform of both sides of Equation (12) gives:

$$\mathscr{L}\left[w_{n+1}(x)\right] = \mathscr{L}\left[w_n(x)\right] + \mathscr{L}\left[\int_0^x \bar{\lambda}(x-\zeta)\left[Lw_n(\zeta) + N\tilde{w}_n(\zeta) - f(\zeta)\right]d\zeta\right], \quad n = 0, 1, 2, \ldots \tag{19}$$

Apply convolution to Equation (19), we get

$$\begin{aligned} \mathscr{L}\left[w_{n+1}(x)\right] &= \mathscr{L}\left[w_n(x)\right] + \mathscr{L}\left[\bar{\lambda}(x) \times \left[Lw_n(x) + N\tilde{w}_n(x) - f(x)\right]\right] \\ &= \mathscr{L}\left[w_n(x)\right] + \mathscr{L}\left[\bar{\lambda}(x)\mathscr{L}\left[Lw_n(x) + N\tilde{w}_n(x) - f(x)\right]\right]. \end{aligned} \tag{20}$$

Optimal value of $\bar{\lambda}(x-\zeta)$ is obtained taking variation with respect to $w_n(x)$ given as;

$$\frac{\delta}{\delta w_n}\mathscr{L}\left[w_{n+1}(x)\right] = \frac{\delta}{\delta w_n}\mathscr{L}\left[w_n(x)\right] + \frac{\delta}{\delta w_n}\mathscr{L}\left[\bar{\lambda}(x)\right]\mathscr{L}\left[Lw_n(x) + N\tilde{w}_n(x) - f(x)\right]. \tag{21}$$

Applying variation with respect to $w_n(x)$ gives:

$$\mathscr{L}\left[\delta w_{n+1}\right] = \mathscr{L}\left[\delta w_n\right] + \delta\mathscr{L}\left[\bar{\lambda}\right]\mathscr{L}\left[w_n\right]. \tag{22}$$

Assume L is linear differential operator with constant coefficients as;

$$L(w) = a_0 w + a_1 w' + a_2 w'' + \cdots + a_{n-2}w^{n-2} + a_{n-1}w^{n-1} + a_n w^n \tag{23}$$

where $a_i's$ are constants. The coefficient contains non-constant terms of the form x^k. The Laplace transform of initial operator term is given as:

$$\mathscr{L}[a_n w^n] = a_n s^n \mathscr{L}[w] - a_n \sum_{k=1}^{n} s^{k-1} w^{n-k}(0). \tag{24}$$

The variation with respect to w is given as:

$$\mathscr{L}[\delta w_{n+1}] = \mathscr{L}[\delta w_n] + \mathscr{L}[\bar{\lambda}]\left[\sum_{k=0}^{n} a_k s^k\right]\mathscr{L}[\delta w_n] = \left[1 + \mathscr{L}[\bar{\lambda}]\left[\sum_{k=0}^{n} a_k s^k\right]\right]\mathscr{L}[\delta w_n]. \tag{25}$$

Extremum condition w_{n+1} needs that δw_{n+1}. Meaning the right hand side of Equation (25) should be set to zero. Hence stationary condition is;

$$\mathscr{L}[\bar{\lambda}] = -\frac{1}{\sum\limits_{k=0}^{n} a_k s^k}. \tag{26}$$

3.3. Application of LVIM to the governing equation

Following the basic principle of LVIM , the governing equation is now analyzed as:

$$\begin{aligned}
\mathscr{L}[w_{n+1}(x)] &= \mathscr{L}[w_n(x)] + \mathscr{L}[\bar{\lambda}]\mathscr{L}\left[\frac{d^4 W_n(X)}{dX^4} - 2\lambda^2 m^2 \pi^2 \frac{d^2 W_n(x)}{dX^2}\right. \\
&\left. -(\Omega^2 - k_w - \lambda^4 m^4 \pi^4)W_n(x) - k_p W_n^3(x)\right] \\
&= \mathscr{L}[w_n(x)] + \mathscr{L}[\bar{\lambda}]\Big[(\lambda^4 m^4 \pi^4 - 2\pi^2 s^2 \lambda^2 m^2 + s^2 - \Omega^2 + k_w)\mathscr{L}[w_n(x)] \\
&-w_n''(0) - s w_n''(0) - s^2 w_n'(0) - s^3 w_n(0) + 2m^2\lambda^2\pi^2 w_n(0) + s w_n(0) - k_p \mathscr{L}[W^3(x)]\Big]. \tag{27}
\end{aligned}$$

Taking variation with respect to $w_n(x)$ on both sides of Equation (27), we get

$$\begin{aligned}
\frac{\delta}{\delta w_n}\mathscr{L}[w_{n+1}(x)] &= \frac{\delta}{\delta w_n}\mathscr{L}[w_n(x)] + \frac{\delta}{\delta w_n}\mathscr{L}[\lambda(\bar{x})]\Big[(\lambda^4 m^4 \pi^4 - 2\pi^2 s^2 \lambda^2 m^2 + s^2 - \Omega^2 + k_w)\mathscr{L}[w_n(x)] \\
&-w_n''(0) - s w_n''(0) - s^2 w_n'(0) - s^3 w_n(0) + 2m^2\lambda^2\pi^2 w_n(0) + s w_n(0) - k_p \mathscr{L}[W^3(x)]\Big]
\end{aligned} \tag{28}$$

Simplifying Equation (28) gives,

$$\begin{aligned}
\mathscr{L}[\delta w_{n+1}] &= \mathscr{L}[\delta w_n] + \mathscr{L}[\bar{\lambda}]\left((\lambda^4 m^4 \pi^4 - 2\pi^2 s^2 \lambda^2 m^2 + s^2 - \Omega^2 + k_w)\mathscr{L}[\delta w_n]\right) \\
&= \mathscr{L}[\delta w_n]\left(1 + \mathscr{L}[\bar{\lambda}](\lambda^4 m^4 \pi^4 - 2\pi^2 s^2 \lambda^2 m^2 + s^2 - \Omega^2 + k_w)\right)
\end{aligned} \tag{29}$$

Extremum condition w_{n+1} needs that $\delta w_{n+1} = 0$. Meaning the right hand side of Equation (29) should be set to zero.

$$1 + \mathscr{L}[\bar{\lambda}](\pi^4 - 2\pi^2 s^2 + s^4) = 0, \;\; \mathscr{L}[\bar{\lambda}] = -\frac{1}{(\pi^4 - 2\pi^2 s^2 + s^4)}. \tag{30}$$

For simplicity we adopt,

$$\mathscr{L}[\bar{\lambda}] = -\frac{1}{s^4}. \tag{31}$$

Substituting Equation (31) into Equation (27) results;

$$\begin{aligned}\mathscr{L}[w_{n+1}(x)] &= \mathscr{L}[w_n(x)] - \mathscr{L}\left[\int_0^x \bar{\lambda}(x-\zeta)\left[\frac{d^4W_n(\zeta)}{d\zeta^4} - 2m^2\pi^2\lambda^2\frac{d^2W_n(\zeta)}{d\zeta^2}\right.\right.\\ &\quad \left.\left.-(\Omega^2 - m^4\pi^4\lambda^4 - k_w)W_n(\zeta) - k_pW_n^3(\zeta)\right]d\zeta\right]\\ &= \mathscr{L}[w_n(x)] - \mathscr{L}\left[\frac{x^3}{6}\right]\mathscr{L}\left[\frac{d^4W_n(\zeta)}{d\zeta^4} - 2m^2\pi^2\lambda^2\frac{d^2W_n(\zeta)}{d\zeta^2}\right.\\ &\quad \left.-(\Omega^2 - m^4\pi^4\lambda^4 - k_w)W_n(x) - k_pW_n^3(x)\right]. \end{aligned} \tag{32}$$

Assuming

$$\Phi_0 = \begin{cases} w(0), & \text{if } L = \frac{d}{dx}\\ w(0) + xw'(0), & \text{if } L = \frac{d^2}{dx^2}\\ w(0) + xw'(0) + \frac{x^2}{2!}w''(0), & \text{if } L = \frac{d^3}{dx^3}\\ w(0) + xw'(0) + \frac{x^2}{2!}w''(0) + \frac{x^3}{3!}w'''(0), & \text{if } L = \frac{d^4}{dx^4}, \end{cases} \tag{33}$$

$$w_0 = w(0) + w'(0)x + \frac{1}{2!}w''(0)x^2 + \frac{1}{3!}w'''(0)x^3. \tag{34}$$

Applying condition 9 at $x = 0$ on Equation (34) gives

$$w_0 = w'(0)x + \frac{1}{3!}w'''(0)x^3 = ax + \frac{\beta}{3!}x^{3!}. \tag{35}$$

Then

$$\mathscr{L}[w_1] = \mathscr{L}[w_0] - \mathscr{L}\left[\frac{x^3}{6}\right]\mathscr{L}\left[\frac{d^4W_0(x)}{dX^4} - 2m^2\pi^2\lambda^2\frac{d^2W_0(x)}{dX^2} - (\Omega^2 - m^4\pi^4\lambda^4 - k_w)W_0(x) - k_pW_0^3(x)\right]. \tag{36}$$

$$\mathscr{L}[w_1] = \frac{\alpha s^2+\beta}{s^4} - \frac{1}{s^4}\left[\frac{(-m^4\pi^4\lambda^4 + \Omega^2 - k_w)(\alpha s^2+\beta)}{s^4} + \frac{k_p(\alpha s^2+\beta)^3}{s^{12}}\right]. \tag{37}$$

Inverse Laplace gives the first iteration:

$$\begin{aligned} w_1 &= \Big[-259459200\pi^4\beta\lambda^4m^4x^2 + \beta^3k_px^{10} - 10897286400\pi^4\alpha\lambda^4m^4 + 630\alpha\beta^2k_px^8\\ &+98280\alpha^2\beta k_px^6 + 3603600\alpha^3k_px^4 + 259459200\Omega^2\beta x^2\\ &-259459200\beta k_wx^2 + 10897286400\Omega^2\alpha - 10897286400\alpha k_w\Big]\frac{x^5}{1307674368000} + \frac{1}{6}x(\beta x^2 + 6\alpha), \end{aligned} \tag{38}$$

$$\mathscr{L}[w_2] = \mathscr{L}[w_1] - \mathscr{L}\left[\frac{x^3}{6}\right]\mathscr{L}\left[\frac{d^4W_1(x)}{dX^4} - 2m^2\pi^2\lambda^2\frac{d^2W_1(x)}{dX^2} - (\Omega^2 - m^4\pi^4\lambda^4 - k_w)W_x - k_pW_1^3(x)\right], \tag{39}$$

$$\begin{aligned}\mathscr{L}[w_2] &= \mathscr{L}[w_1] - \frac{1}{s^4}\Bigg[(\Omega^2 - m^4\pi^4\lambda^4 - k_w)\Bigg(-\frac{\pi^4\beta\lambda^4m^4}{s^8} + \frac{\Omega^2\beta}{s^8} - \frac{\beta k_w}{s^8} - \frac{\pi^4\alpha\lambda^4m^4}{s^6} + \frac{\Omega^2\alpha}{s^6} - \frac{\alpha k_w}{s^6}\\ &+\frac{3\alpha^2\beta k_p}{s^{12}} + \frac{\alpha^3k_p}{s^{10}} + \frac{3\alpha\beta^2k_p}{s^{14}} + \frac{\beta^3k_p}{s^{16}} + \frac{\beta}{s^4} + \frac{\alpha}{s^2}\Bigg) + k_p\Bigg(-\frac{\pi^4\beta\lambda^4m^4}{s^8} + \frac{\Omega^2\beta}{s^8} - \frac{\beta k_w}{s^8} - \frac{\pi^4\alpha\lambda^4m^4}{s^6} + \frac{\Omega^2\alpha}{s^6}\\ &-\frac{\alpha k_w}{s^6} + \frac{3\alpha^2\beta k_p}{s^{12}} + \frac{\alpha^3k_p}{s^{10}} + \frac{3\alpha\beta^2k_p}{s^{14}} + \frac{\beta^3k_p}{s^{16}} + \frac{\beta}{s^4} + \frac{\alpha}{s^2}\Bigg)\Bigg]. \end{aligned} \tag{40}$$

Inverse Laplace gives the second iteration:

$$\begin{aligned} w_2 \;=\; & \Big[-14244993572210851712262988263587840000000\pi^{12}\beta^3 k_p\lambda^{12}m^{12}x^{22} \\ & +450332877744475442196480000\pi^8\beta^5 k_p^2\lambda^8 m^8 x^{30} \\ & -2999995646286053706025853283115991040000000\pi^{12}\alpha\beta^2 k_p\lambda^{12}m^{12}x^{20} \\ & -77022858528000\pi^4\beta^7 k_p\lambda^4 m^4 x^{38} + \ldots \Big] x^5 + \frac{1}{3}x(\beta x^2 + 6\alpha) \\ & + \Big[-259459200\pi^4\beta\lambda^4 m^4 x^2 + \beta^3 k_p x^{10} - 10897286400\pi^4\alpha\lambda^4 m^4 \\ & +630\alpha\beta^2 k_p x^8 98280\alpha^2\beta k_p x^6 - 3603600\alpha^3 k_p x^4 + 259459200\Omega^2\beta x^2 - 259459200\beta k_w x^2 \\ & +10897286400\Omega^2\alpha - 10897286400\alpha_w \Big] \frac{x^5}{1307674368000}, \end{aligned} \tag{41}$$

The same approach is continued till frequency parameter Ω obtained converges. Substituting boundary condition at $x = 1$ to find the unknowns introduced results into simultaneous equation.

Table 1. Parameters for validation of the model

Pasternak foundation	Winkler foundation	Poisson ratio	Integer	Aspect ratio
K_w	K_p	v	m	λ
0	0	0.3	1	1

Table 1 contains parameters for validation of the approach to ascertain the correctness of the results.

$$\begin{aligned} \psi_{11}^n(\Omega)w_0 + \psi_{12}^n(\Omega)w_2 \\ \psi_{21}^n(\Omega)w_0 + \psi_{22}^n(\Omega)w_2. \end{aligned} \tag{42}$$

The polynomials are represented as $\psi_{11}, \psi_{12}, \psi_{21}$ and ψ_{22}. Equation (42) can be written in matrix form as:

$$\begin{bmatrix} \psi_{11}^n(\Omega) & \psi_{12}^n(\Omega) \\ \psi_{21}^n(\Omega) & \psi_{22}^n(\Omega) \end{bmatrix} \begin{bmatrix} w_0 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{43}$$

The following Characteristic determinant is obtained applying the non-trivial condition

$$\begin{bmatrix} \psi_{11}^n(\Omega) & \psi_{12}^n(\Omega) \\ \psi_{21}^n(\Omega) & \psi_{22}^n(\Omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{44}$$

Solving Equation (44) gives the natural frequencies. Substitute the result obtained into Equation (43), we get

$$\begin{bmatrix} \frac{245431}{40824} & \frac{32120}{19453} \\ \frac{-181919}{3773} & \frac{-125273}{9460} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{45}$$

Setting $\alpha = 1$ and find β

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_1 = \begin{bmatrix} 1 \\ -3.641035489 \end{bmatrix} \tag{46}$$

Same procedure is repeated for other modes.

$$w(x) = \left[\frac{4.27 \times 10^{58}\pi^8 x^4 - 2.36 \times 10^{60}x^6 - 3.34 \times 10^{62}x^4 - 4.58 \times 10^{62}x^2 + 5.28 \times 10^{63}}{1.55 \times 10^{64}}\right]x^5 - \frac{1}{3}\left[-\frac{47399}{13018}x^2 - 6\right]x - \left[\frac{3.86 \times 10^8 x^2 - 4.45 \times 10^9}{1.30 \times 10^{10}}\right]x^5 \quad (47)$$

The following convergence criterion may be used

$$\frac{|\Omega_j^{(i)} - \Omega_j^{(i-1)}|}{\Omega_j^{(i)}} \leq \varepsilon, \quad j = 1, 2, 3, \ldots, n \quad (48)$$

where ε is the tolerance parameter taken to be 0.0001 for this study, Ω_j represents the Eigenvalue.

The iteration converges at third iteration for first mode frequency parameter.

4. Results and discussion

The solution of Laplace and Variation iteration method is presented here. Table 2 shows the comparison of present results to that of previously published work. It is realized from the Table 2 that, good agreements is achieved with that of the past results.The fundamental modal shape of the thin rectangular plate are shown in Figures 3, 4, 5 and it is observed that the shape obeys classical plate theory. Also, Table 3 shows different deflection values of transverse displacement for the first three mode frequency parameters of SC, SS and SF boundary condition considered. Table 4 shows the convergence study, it is observed that the fundamental natural frequency converges at the third iteration while higher modes are obtained by increasing the number of iterations. This phenomenon is peculiar to vibration problem.

Table 2. Showing validation of results

Edge Condition/Dimensionless	Simply-supported (SS)		Simply supported -Clamped (SC)		Simply-supported -Free (SF)	
Natural frequency	Bhat *et al.* [2]	Present	Leissa [38]	Present	Leissa [38]	Present
$\Omega 1$	19.7392	19.7434	23.6463	23.6486	11.7195	11.7606

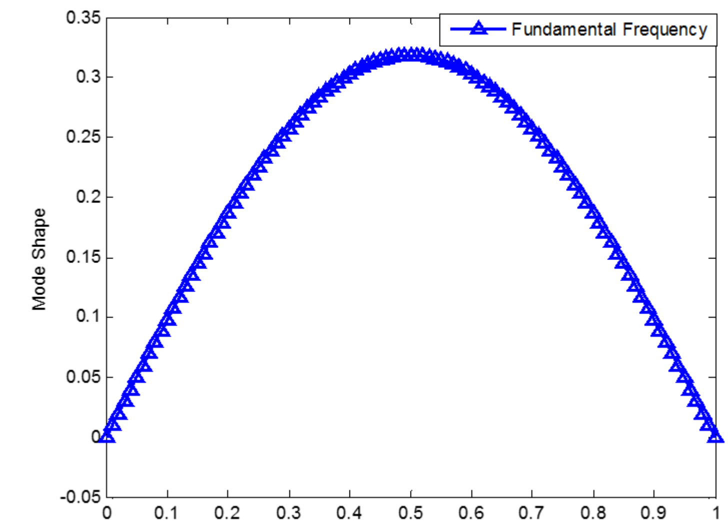

Figure 3. Fundamental mode shape of simply supported condition at both edges

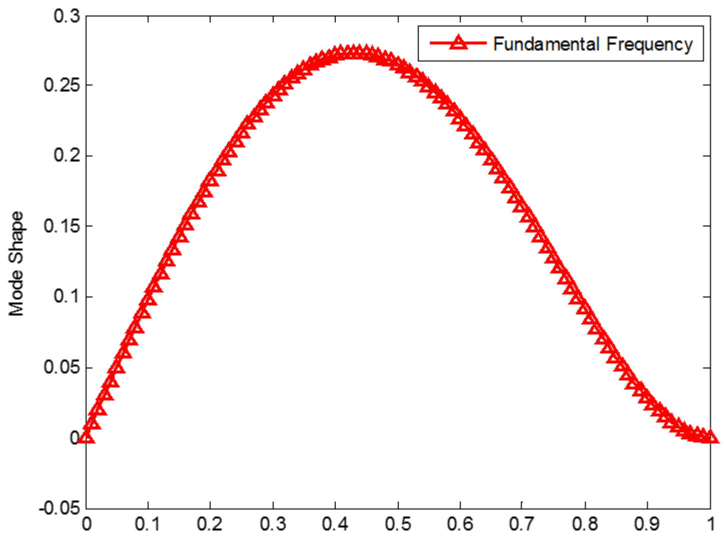

Figure 4. Fundamental mode shape of simply supported with clamped edge condition

It is also observed that, the presence of elastic foundation and aspect ratio has no significant changes on the mode shape of the rectangular plate. Since dimensionless analysis is carried out, the results are valid for all thin plates. Table 4 shows that the value of frequency parameters Ω decreases in the order of $SC \geq SS \geq SF$.

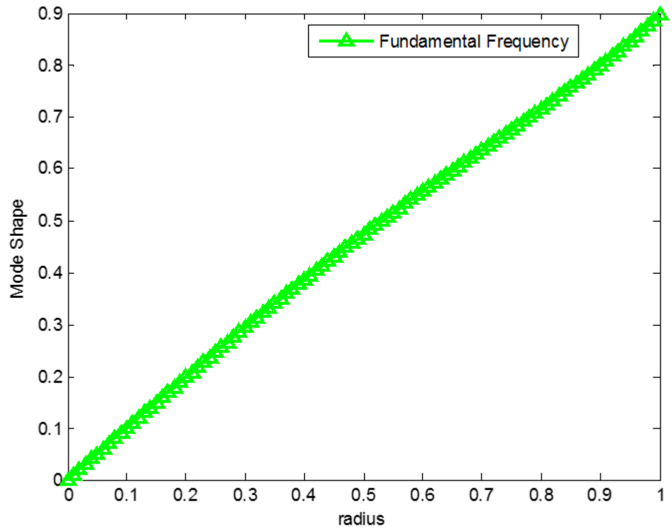

Figure 5. Fundamental mode shape of Simply Supported Condition at one edge and free at other edge

4.1. Effect of foundation parameter on natural frequency

Table 5 illustrates the impact of foundation parameter on natural frequency. It is clear from the Figures 6, 7 and 8 that the foundation parameter has impact on natural frequency, increasing values of the foundation parameter increases the natural frequency. This satisfies the principle of classical vibration. Stiffness increment results to natural frequency increment. This also corroborated with finding reported in [38]. The effect of increase in natural frequency is much significant in higher values of the elastic foundation.

4.2. Effect of variation of aspect ratio on natural frequency

The influence of aspect ratio on natural frequency are shown in Table 6 and Figures 9, 10, 11 respectively. It is shown that, the natural frequency increase with increases in aspect ratio. This is because, the plate becomes more stiff as the aspect ratio increases resulting in the natural frequency increases.

Table 3. Results of different deflection values

Transverse displacement	SF	SS	SC	SF	SS	SC	SF	SS	SC
		Ω1			Ω2			Ω3	
w[0]	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
w[0.05]	0.0500	0.0498	0.0497	0.0496	0.0492	0.0490	0.0489	0.0482	0.0479
w[0.10]	0.0998	0.0984	0.0978	0.0970	0.0935	0.0921	0.0916	0.0858	0.0837
w[0.15]	0.1492	0.1445	0.1425	0.1400	0.1288	0.1240	0.1224	0.1048	0.0983
w[0.20]	0.1981	0.1871	0.1825	0.1767	0.1514	0.1410	0.1376	0.1009	0.0881
w[0.25]	0.2464	0.2251	0.2163	0.2054	0.1592	0.1409	0.1351	0.0750	0.0556
w[0.30]	0.2938	0.2575	0.2429	0.2248	0.1514	0.1237	0.1153	0.0328	0.0091
w[0.35]	0.3403	0.2836	0.2615	0.2340	0.1288	0.0915	0.0807	-0.0166	-0.0397
w[0.40]	0.3857	0.3027	0.2715	0.2323	0.0935	0.0482	0.0358	-0.0623	-0.0784
w[0.45]	0.4300	0.3144	0.2727	0.2199	0.0492	-0.0011	-0.0134	-0.0945	-0.0973
w[0.50]	0.4732	0.3183	0.2652	0.1971	0.0000	-0.0506	-0.0607	-0.1061	-0.0915
w[0.55]	0.5153	0.3144	0.2496	0.1647	-0.0492	-0.0945	-0.0998	-0.0946	-0.0624
w[0.60]	0.5564	0.3027	0.2266	0.1239	-0.0935	-0.1275	-0.1254	-0.0624	-0.0173
w[0.65]	0.5967	0.2836	0.1976	0.0760	-0.1288	-0.1462	-0.1340	-0.0167	0.0326
w[0.70]	0.6363	0.2575	0.1641	0.0227	-0.1514	-0.1487	-0.1240	0.0327	0.0751
w[0.75]	0.6757	0.2251	0.1279	-0.0345	-0.1592	-0.1356	-0.0958	0.0750	0.0999
w[0.80]	0.7154	0.1871	0.0914	-0.0939	-0.1514	-0.1095	-0.0520	0.1009	0.1018
w[0.85]	0.7561	0.1445	0.0571	-0.1542	-0.1288	-0.0754	0.0036	0.1048	0.0821
w[0.90]	0.7987	0.0984	0.0281	-0.2148	-0.0935	-0.0400	0.0664	0.0859	0.0487
w[0.95]	0.8444	0.0498	0.0078	-0.2753	-0.0492	-0.0117	0.1325	0.0482	0.0155
w[1.00]	0.8948	-9.0000	-9.0000	-0.3364	0.0000	0.0000	0.1995	0.0000	0.0000

Table 4. Showing convergence study of the results

Edge Condition /Dimensionless Natural frequency	Iteration	(SS)		(SC)		(SF)	
		Bhat *et al.* [39]	Present	Leissa [38]	Present	Leissa [38]	Present
Ω1	N3	19.7392	19.7434	23.6463	23.6486	11.7195	11.7606
Ω1	N4	19.7392	19.9574	23.6463	23.8905	11.7195	11.7445
Ω2		49.3481	49.3271	58.6465	58.6240	27.7563	27.7563
Ω1	N6	19.7392	19.7418	23.6463	23.6487	11.7195	11.6855
Ω2		49.3481	49.0637	58.6465	58.3220	27.7563	27.6965
Ω1	N7	19.7392	19.7394	23.6463	23.6465	11.7195	11.6846
Ω2		49.3481	49.3271	58.6465	58.6240	27.7563	27.7563

Table 5. Variation elastic foundation coefficient on natural frequency

Edge Condition	Natural frequency	kw=5	kw=15	kw=45	kw=120	kw=200	kw=250
SS	Ω1	19.865457	20.115575	20.847934	22.575127	24.282429	25.291033
	Ω2	49.398659	49.49977	49.801882	50.549258	51.334467	51.81918
SC	Ω1	23.751809	23.961395	24.579431	26.060476	27.552648	28.445534
	Ω2	58.688978	58.774102	59.02877	59.660676	60.327409	60.7404
SF	Ω1	11.896571	12.309687	13.473248	16.016504	18.34471	19.660326
	Ω2	27.846269	28.025251	28.555467	29.839817	31.151479	31.94393

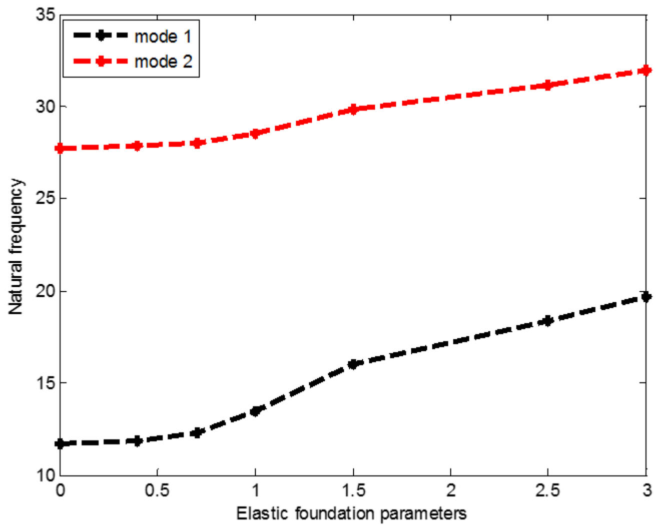

Figure 6. Variation of elastic foundation parameter on SF edge condition

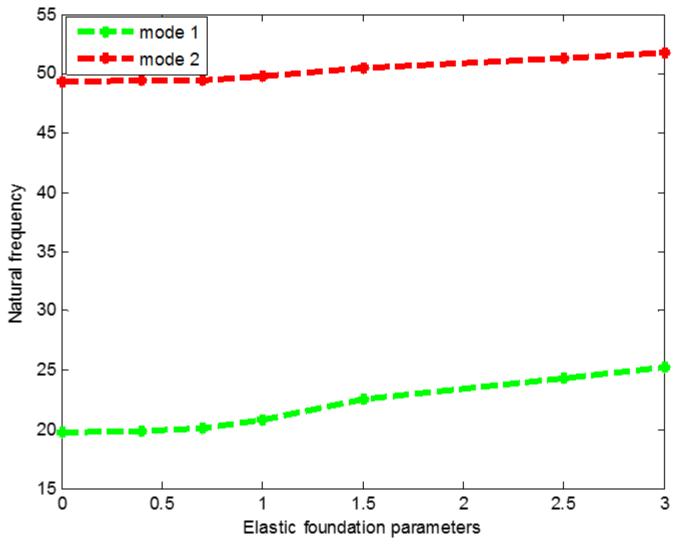

Figure 7. Variation of elastic foundation parameter on SS-edge condition

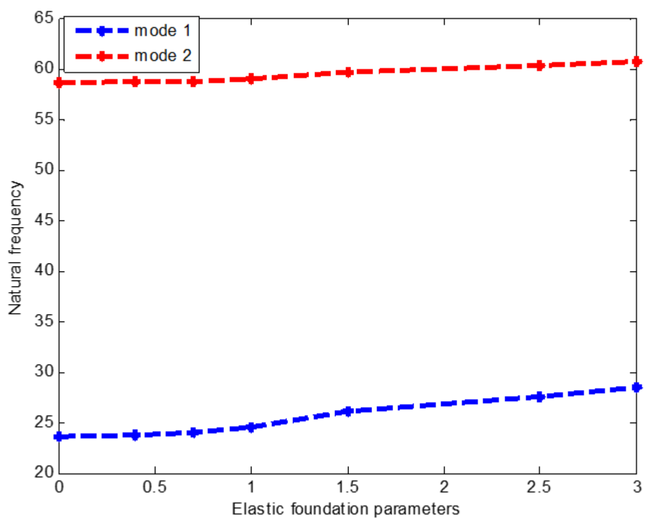

Figure 8. Variation of elastic foundation parameter on SC-edge condition

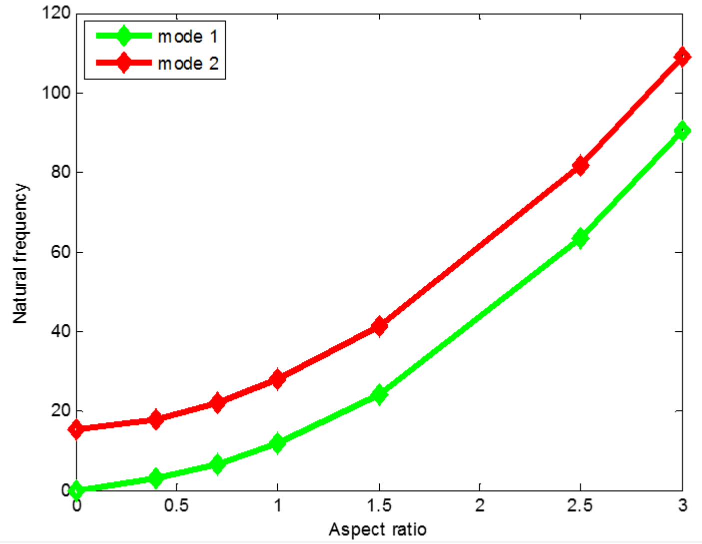

Figure 9. Variation of Aspect ratio on SF edge condition

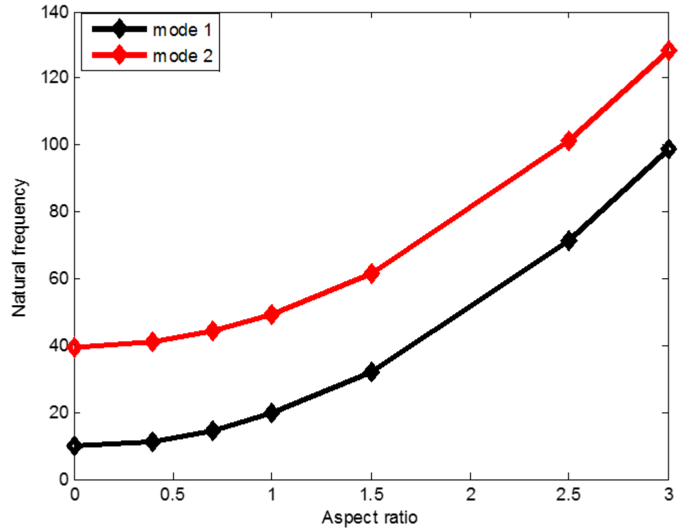

Figure 10. Variation of Aspect ratio on SS edge condition

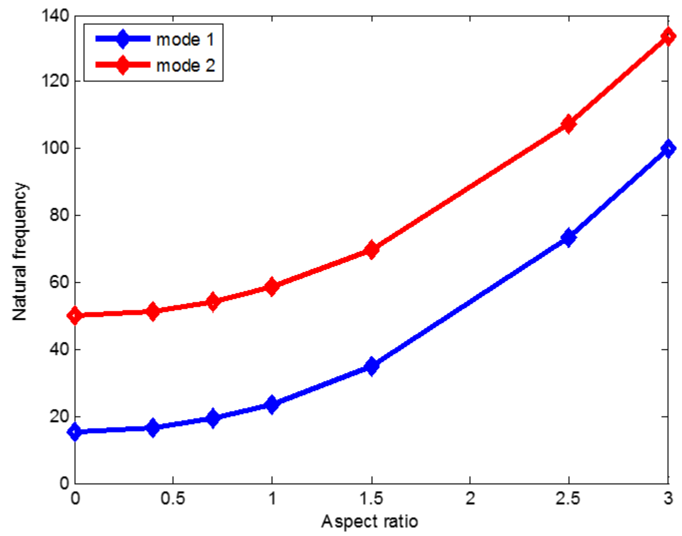

Figure 11. Variation of Aspect ratio on SC edge condition

Table 6. Variation of aspect ratio on natural frequency

Edge Condition	Natural frequency	$\lambda = 0.4$	$\lambda = 0.7$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2.5$	$\lambda = 3.0$
SS	Ω1	11.448741	14.705711	19.739209	32.076214	71.554632	98.696055
	Ω2	41.057554	44.314525	49.348024	61.685024	101.16349	128.3048
SC	Ω1	16.627624	19.277232	23.64632	35.051125	73.438926	100.26994
	Ω2	51.326727	54.171092	58.646365	69.912813	107.42	133.79231
SF	Ω1	3.0081474	6.562375	11.684537	24.010127	63.28683	90.296225
	Ω2	17.636147	21.826148	27.756345	41.173975	81.606845	108.92412

5. Conclusion

In this study, the dynamic analysis of isotropic rectangular plates resting on Winkler and Pasternak foundations is analyzed. The governing equation is transform to nonlinear ordinary differential equation using Galerkin method of separation. The nonlinear ordinary differential equations have been solved using Laplace transform and variation of iteration method. The accuracies of the obtained analytical solutions were ascertained with the results obtained by earlier researcher. The obtained analytical solutions were used to examine the effects of foundation parameter, aspect ratio. The rate of convergence is increased with the introduction of exact method for analyzing the linear part of the governing equation while the remaining part are treated with variation of iteration method, practical applications of the study are base plate of tower, steel hinged steel column structures and culvert covers. From the parametric studies, the following observations were established:

1. Increase in elastic foundation parameter increases the natural frequency.
2. Increase in aspect ratio increases the natural frequency.
3. Increasing the combine elastic foundation parameters increases the natural frequency.
4. Accurate higher mode frequency can be obtained with increase in number of iterations.
5. SF boundary condition has the least value of frequency parameter followed by SS edge condition.
6. The effect of increase in natural frequency is much significant in higher value of the elastic foundation.

Abbreviations

Abbreviations	Nomenclature
a	Length of the plate
b	Width of the plate
C	Clamped edge plate
E	Young's modulus
F	Free edge support
S	Simply supported edge
d/dx	Differential operator
w	Dynamic deflection
X	space coordinate along the length of thin plate Symbol
h	plate thickness
ρ	Mass density
D	Modulus of elasticity
Ω	natural frequency

Acknowledgments: The author expresses sincere appreciation to the management of University of Lagos, Nigeria, for providing material supports and good environment for this work.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

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