SOME MULTI-STEP ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS-I

MUHAMMAD SAQIB[1], MUHAMMAD IQBAL

ABSTRACT. In this paper, we suggest and analyze two new algorithm of fourth and fifth order convergence. We rewrite nonlinear equation as an equivalent coupled system and then use modified decomposition technique to develop our algorithms. Convergence analysis of newly introduced algorithms has been discussed. To see efficiency and performance of these algorithms, we have made comparison of these algorithms with some well known algorithms existing in literature.

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1. Introduction

To find the root of nonlinear equation of the form $f(x) = 0$, is the oldest and basic problem in numerical analysis. Newton's method is one of oldest and most powerful formula to approximate the root of nonlinear equations. It has second order convergence. Many modifications in Newton's method have been made to increase its convergence order using various techniques. Recently, many iterative methods with higher order convergence have been established using different techniques like Taylor's series, Adomain decomposition, homotopy, modified homotopy, decomposition, modified decomposition, interpolation, quadrature rules and their many modifications. see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and references therein. Chun [3] introduced some multi-step iterative methods using Adomain decomposition. These method requires higher order derivatives. Later on, Noor

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[1] Corresponding Author

[4, 5] established some multi-step iterative methods that do not require higher order derivative using a different technique.

In this paper, some numerical methods based on decomposition technique are proposed for solving algebraic nonlinear equations. For this purpose, we write nonlinear equations as an equivalent system of equations and use technique introduced by Gejji and Jafari [8]. Recently, several iterative methods have been suggested by writing nonlinear equations as an equivalent system of equations and then using different techniques. In section 3, we give detailed proof regarding convergence of our newly established iterative methods. In the last section, numerical results are given to make comparison of these algorithm with some classical methods.

2. Iterative methods

Consider the nonlinear equation

$$f(x) = 0; \quad x \in \mathbb{R}, \tag{1}$$

assume α is a simple root of (1) and γ is an initial guess sufficiently close to α. We can write (1) as following coupled system;

$$f(\gamma) + (x - \gamma)f'(\frac{x + \gamma}{2}) + g(x) = 0, \tag{2}$$

$$g(x) = f(x) - f(\gamma) - (x - \gamma)f'(\frac{x + \gamma}{2}). \tag{3}$$

From Eq. (2), we have

$$x = \gamma - \frac{f(\gamma)}{f'(\frac{x+\gamma}{2})} - \frac{g(x)}{f'(\frac{x+\gamma}{2})}.$$

Let

$$x = c + N(x), \tag{4}$$

where

$$c = \gamma - \frac{f(\gamma)}{f'(\frac{x+\gamma}{2})}, \tag{5}$$

and

$$N(x) = -\frac{g(x)}{f'(\frac{x+\gamma}{2})}. \tag{6}$$

Now, we construct a sequence of higher-order iterative methods by applying modified decomposition technique introduced by Gejji and Jafari [8]. This technique consists a solution of equation (3) that can be written in the form of infinite series:

$$x = \sum_{i=0}^{\infty} x_i, \tag{7}$$

and using Gejji and Jafari [8] technique, we decompose the nonlinear function N as;

$$N(\sum_{i=0}^{\infty} x_i) = N(x_0) - \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} x_j) - N(\sum_{j=0}^{i-1} x_j)\}. \qquad (8)$$

Thus from Eqs. (4), (7) and (8) we have

$$\sum_{i=0}^{\infty} x_i = x_0 + N(x_0) + \sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i} x_j) - N(\sum_{j=0}^{i-1} x_j)\}.$$

So we have iteration scheme as

$$
\begin{aligned}
x_0 &= c \\
x_1 &= N(x_0) \\
x_2 &= N(x_0 + x_1) - N(x_0) \\
&\quad . \\
&\quad . \\
&\quad . \\
x_{n+1} &= N(x_0 + x_1 + ... + x_n) - N(x_0 + x_1 + ... + x_{n-1}), \ n = 1, 2, ...
\end{aligned}
$$

When

$$
\begin{aligned}
x &\approx x_0 \\
&= c \\
&= \gamma - \frac{f(\gamma)}{f'(\frac{x+\gamma}{2})}.
\end{aligned}
$$

From above relation, we can write the algorithm as follows;

Algorithm 2.1: For any initial value x_0, we compute the approximation solution x_{n+1}, by the iteration scheme;

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'(\frac{x_n+y_n}{2})}.
\end{aligned}
$$

This algorithm has third order convergence and has been established by Frontini and Sormani [1].
When

$$
\begin{aligned}
x &\approx x_0 + x_1 \\
&= \gamma - \frac{f(\gamma)}{f'(\frac{x+\gamma}{2})} - \frac{g(x_0)}{f'(\frac{x+\gamma}{2})}.
\end{aligned}
$$

From Eq. (3), we see that

$$g(x_0) = f(x_0),$$

by substituting in above, we get

$$x = \gamma - \frac{f(\gamma)}{f'(\frac{x+\gamma}{2})} - \frac{f(x_0)}{f'(\frac{x_0+\gamma}{2})},$$

Using this, we suggest the following three step iterative methods for solving Eq. (1) as follows;

Algorithm 2.2: For any initial value x_0, we compute the approximation solution x_{n+1}, by the iteration scheme;

$$\text{Predictor Steps;} \qquad y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{f(x_n)}{f'(\frac{x_n+y_n}{2})}$$

$$\text{Corrector Step;} \qquad x_{n+1} = z_n - \frac{f(z_n)}{f'(\frac{x_n+z_n}{2})}$$

When

$$\begin{aligned}
x &\approx x_0 + x_1 + x_2 \\
&= x_0 + N(x_0 + x_1) \\
&= x_0 - \frac{g(x_0 + x_1)}{f'(\frac{x_0+x_1+\gamma}{2})} \\
&= x_0 - \frac{f(x_0 + x_1) + f(x_0)}{f'(\frac{x_0+x_1+\gamma}{2})}
\end{aligned}$$

From this relation, we formulate four step iterative method for solving Eq. (1) as follows;

Algorithm 2.3: For any initial value x_0, we compute the approximation solution x_{n+1}, by the iteration scheme;

$$\text{Predictor Steps;} \qquad y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{f(x_n)}{f'(\frac{x_n+y_n}{2})}$$

$$u_n = z_n - \frac{f(z_n)}{f'(\frac{x_n+z_n}{2})}$$

$$\text{Corrector Step;} \qquad x_{n+1} = z_n - \frac{f(u_n) + f(z_n)}{f'(\frac{x_n+u_n}{2})}$$

3. Convergence Analysis

In this section, we discuss in detail the convergence analysis of Algorithm 2.2 and 2.3 established above.

Theorem 3.1. *For an open interval $I \subset \mathbb{R}$ let $f : I \to \mathbb{R}$ and $\alpha \in I$ be simple zero of $f(x) = 0$. If f is differentiable and x_0 is sufficiently close to α then three-step iterative method defined by Algorithm 2.2 has fourth order convergence.*

Proof. Expanding $f(x)$ by Taylor's series about α, we have

$$f(x_n) = f'(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + ... \tag{9}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, and $e_n = x_n - \alpha$.

$$f'(x_n) = f'(\alpha)(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + ... \tag{10}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

From above equations,we get

$$y_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 4c_2^3)e_n^4 + (4c_5 - 10c_2 c_4$$
$$- 6c_3^2 + 20c_3 c_2^2 - 8c_2^4)e_n^5 + ... \tag{11}$$

Now, expanding $f'(\frac{x_n + y_n}{2})$ by Taylor's series about α,

$$f'(\frac{x_n + y_n}{2}) = f'(\alpha)\{1 + c_2 e_n + (c_2^2 + \frac{3}{4}c_3)e_n^2 + (\frac{7}{2}c_2 c_3 - 2c_2^3 + \frac{1}{2}c_4)e_n^3$$
$$+ (3c_3^2 - \frac{37}{4}c_3 c_2^2 + \frac{3}{2}c_2 c_4 + 4c_2^4 + \frac{5}{16}c_5)e_n^4 + ...\} \tag{12}$$

Dividing Eq. (9) by Eq. (12)

$$\frac{f(x_n)}{f'(\frac{x_n + y_n}{2})} = e_n + (-c_2^2 + \frac{1}{4}c_3)e_n^3 + (-\frac{15}{4}c_2 c_3 + 3c_2^3 + \frac{1}{2}c_4)e_n^4$$
$$+ (-\frac{51}{16}c_3^2 + \frac{22}{7}c_3 c_2^2 - 2c_2 c_4 - 6c_2^4 + \frac{16}{11}c_5)e_n^5 + ... \tag{13}$$

Now

$$z_n = x_n - \frac{f(x_n)}{f'(\frac{x_n + y_n}{2})}$$

Putting values in above, we get

$$z_n = \alpha + (c_2^2 - \frac{1}{4}c_3)e_n^3 + (\frac{15}{4}c_2 c_3 - 3c_2^3 - \frac{1}{2}c_4)e_n^4$$
$$+ (\frac{51}{16}c_3^2 - \frac{22}{7}c_3 c_2^2 + 2c_2 c_4 - 6c_2^4 + \frac{16}{11}c_5)e_n^5 + ... \tag{14}$$

Expanding $f(z_n)$ and $f'(\frac{x_n + z_n}{2})$ by Taylor's series about α,

$$f(z_n) = f'(\alpha)\{(c_2^2 - \frac{1}{4}c_3)e_n^3 + (\frac{15}{4}c_2 c_3 - 3c_2^3 - \frac{1}{2}c_4)e_n^4$$

$$+ (\frac{51}{16}c_3^2 - \frac{22}{7}c_3c_2^2 + 2c_2c_4 - 6c_2^4 + \frac{16}{11}c_5)e_n^5 + ...\} \tag{15}$$

$$f'(\frac{x_n + z_n}{2}) = f'(\alpha)\{1 + c_2e_n + \frac{3}{4}c_3e_n^2 + (c_2^3 - \frac{1}{4}c_2c_3 + \frac{1}{2}c_4)e_n^3 + (\frac{21}{4}c_3c_2^2 - 3c_2^3$$

$$- \frac{1}{2}c_2c_4 + \frac{5}{16}c_5 - \frac{3}{8}c_3^2)e_n^4 + ...\} \tag{16}$$

Dividing (15) by Eq. (16), we have

$$\frac{f(z_n)}{f'(\frac{x_n+z_n}{2})} = (c_2^2 - \frac{1}{4}c_3)e_n^3 + (4c_2c_3 - 4c_2^3 - \frac{1}{2}c_4)e_n^4 + (\frac{27}{8}c_3^2 - \frac{73}{4}c_3c_2^2$$

$$+ \frac{5}{2}c_2c_4 + 10c_2^4 - \frac{11}{16}c_5)e_n^5 + ...$$

Corrector step of Algorithm 2.2 is

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(\frac{x_n+z_n}{2})}$$

By substitution and simplification, we have

$$x_{n+1} = \alpha + (4c_2c_3 - 4c_2^3 - \frac{1}{2}c_4)e_n^4 + (\frac{27}{8}c_3^2 - \frac{73}{4}c_3c_2^2 + \frac{5}{2}c_2c_4 + 10c_2^4 - \frac{11}{16}c_5)e_n^5 + ...$$
$$\tag{17}$$

Hence Algorithm 2.2 has fourth order convergence. □

Theorem 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I and $\alpha \in I$ be simple zero of $f(x) = 0$. If f is differentiable and x_0 is sufficiently close to α then three-step iterative method defined by Algorithm 2.3 has fifth order convergence and satisfies the error equation*

$$x_{n+1} = \alpha + (-\frac{1}{4}c_3c_2^2 + c_2^4)e_n^5 + O(e_n^6), \ \ and \ e_n = x_n - \alpha.$$

Proof. From Eq. (17), we have

$$u_n = \alpha + (4c_2c_3 - 4c_2^3 - \frac{1}{2}c_4)e_n^4 + (\frac{27}{8}c_3^2 - \frac{73}{4}c_3c_2^2 + \frac{5}{2}c_2c_4 + 10c_2^4 - \frac{11}{16}c_5)e_n^5 + ...,$$

Expanding $f(u_n)$ and $f'(\frac{x_n+u_n}{2})$ by Taylor's series about α,

$$f(u_n) = f'(\alpha)\{(4c_2c_3 - 4c_2^3 - \frac{1}{2}c_4)e_n^4 + (\frac{27}{8}c_3^2 - \frac{73}{4}c_3c_2^2 + \frac{5}{2}c_2c_4 + 10c_2^4 - \frac{11}{16}c_5)e_n^5 + ...\},$$
$$\tag{18}$$

$$f'(\frac{x_n+u_n}{2}) = f'(\alpha)\{1 + c_2e_n + \frac{3}{4}c_3e_n^2 + \frac{1}{2}c_4e_n^3 + (c_2^3 - \frac{1}{4}c_3c_2^2 + \frac{5}{16}c_4)e_n^4 + ...\}. \tag{19}$$

Now

$$x_{n+1} = z_n - \frac{f(u_n) + f(z_n)}{f'(\frac{x_n+u_n}{2})},$$

by substituting values and simplifying, we have

$$x_{n+1} = \alpha + (-\frac{1}{4}c_3c_2^2 + c_2^4)e_n^5 + O(e_n^6).$$

Hence Algorithm 2.3 has fifth order convergence.   □

4. Applications

In this section, we present some numerical examples to examine the validity and efficieny of our newly devolped algorithms namely, Algorithm 2.2 and Algorithm 2.3. We also provide comparison of these algorithms with Newton's method(NM), Abbasbandy's method (AM), Householder's method(HHM), Halley's method (HM), Chun's method (CM) [3] and Noor's method(NR)[4]. We use $\epsilon = 10^{-25}$. We use the following stopping criteria:

$$(i) \qquad |x_n - x_{n-1}| \; < \; \in$$
$$(ii) \qquad |f(x_n)| \; < \; \in$$

We consider the following nonlinear scalar equations for comparison:

$$\begin{aligned}
f_1(x) &= \sin^2 x - x^2 + 1 = 0 \\
f_2(x) &= x^2 - e^x - 3x + 2 = 0 \\
f_3(x) &= \cos x - x = 0 \\
f_4(x) &= (x-1)^3 - 1 = 0 \\
f_5(x) &= x^3 - 10 = 0 \\
f_6(x) &= e^{x^2 + 7x - 30} - 1 = 0.
\end{aligned}$$

Comparison Table

Examples	Iterations	x_n	$f(x_n)$
$f_1,\ x_0 = 1$			
NM	7	1.40449164831534122635086	$-1.05e^{-50}$
AM	5	1.40449164831534122635086	$-5.82e^{-54}$
HM	4	1.40449164831534122635086	$-5.45e^{-63}$
CM	5	1.40449164831534122635086	$-2.01e^{-64}$
HHM	6	1.40449164821534122603508	$1.82e^{-25}$
NR	5	1.40449164821534122603508	$9.23e^{-26}$
Alg. 2.2	3	1.40449164821534122603508	$3.40e^{-25}$
Alg. 2.3	3	1.40449164821534122603509	$2.32e^{-44}$
$f_2,\ x_0 = 2$			
NM	6	0.25753028543986076045367	$2.94e^{-55}$
AM	5	0.25753028543986076045367	$1.03e^{-63}$
HM	5	0.25753028543986076045367	0
CM	4	0.25753028543986076045367	$1.05e^{-62}$
HHM	5	0.25753028543986076045367	$-6.01e^{-25}$
NR	5	0.25753028543986076045367	$1.08e^{-23}$
Alg. 2.2	3	0.25753028543986093497593	$1.72e^{-69}$
Alg.2.3	2	0.25753028543986093497593	$1.01e^{-29}$

Examples	Iterations	x_n	$f(x_n)$
$f_3,\ x_0 = 1.7$			
NM	5	0.73908513321516064165537	$-2.04e^{-31}$
AM	4	0.73908513321516064165537	$-7.13e^{-47}$
HM	4	0.73908513321516064165537	$-5.04e^{-58}$
CM	4	0.73908513321516064165537	0
HHM	4	0.73908513321516064165537	$3.91e^{-17}$
NR	4	0.73908513321516064562885	$6.66e^{-18}$
Alg. 2.2	3	0.73908513321516064165531	$3.21e^{-50}$
Alg.2.3	3	0.73908513321516064165531	$1.21e^{-100}$
$f_4,\ x_0 = 1.5$			
NM	8	2.000000000000000000001234	$2.05e^{-42}$
AM	5	2	0
HM	5	2	0
CM	5	2	0
HHM	7	2.000000000000000000000828	$2.47e^{-22}$
NR	5	2.000000000000000000000302	$1.12e^{-24}$
Alg. 2.2	4	2.000000000000000000000007	$1.83e^{-44}$
Alg. 2.3	4	2.000000000000000000000000	$5.07e^{-27}$
$f_5,\ x_0 = 1.5$			
N.M.	7	2.1544346900318837217592335	$2.05e^{-54}$
A.M.	5	2.1544346900318837217592335	$-5.01e^{-63}$
H.M.	5	2.1544346900318837217592335	$-5.03e^{-62}$
C.M.	5	2.1544346900318837217592335	$-5.01e^{-64}$
H.H.M.	6	2.1544346900318837217592293	$7.86e^{-27}$
NR.M.	4	2.1544346900318837217592292	$8.98e^{-24}$
Alg. 2.2	3	2.1544346900318837217592294	$2.00e^{-27}$
Alg. 2.3	3	2.1544346900318837217592293	$1.33e^{-50}$
$f_6,\ x_0 = 3.5$			
N.M.	13	3.0000000000000000000000000	$1.52e^{-48}$
A.M.	7	3.0000000000000000000000000	$-4.32e^{-48}$
H.M.	8	3.0000000000000000000000000	$2.01e^{-62}$
C.M.	8	3.0000000000000000000000000	$2.05e^{-62}$
H.H.M.	12	3.0000000000000000000000000	$5.47e^{-25}$
NR.M.	6	3.0000010285594873990664	$1.34e^{-06}$
Alg. 2.2	4	3.0000000000000000000000000	$4.76e^{-30}$
Alg. 2.3	4	3.0000000000000000000000000	$1.99e^{-102}$

5. Conclusion

We have established two new algorithms of fourth and fifth order convergence by using modified decomposition technique to coupled system. We have discussed convergence analysis of these newly established algorithms. Computational comparison of these algorithms has been made with some well known methods for solving nonlinear equations. From numerical table, we see that our new methods

converge fast to the true solution of equations and are comparable with some classical methods.

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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Muhammad Saqib

Department of Mathematics, Lahore Leads University, Lahore-54590, Pakistan.
e-mail: saqib270@yahoo.com

Muhammad Iqbal

Department of Mathematics, Lahore Leads University, Lahore-54590, Pakistan.
e-mail: iqbal66dn@yahoo.com