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Necessary and Sufficient Conditions for L^1 - convergence of Cosine Trigonometric Series

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Abstract

We obtain a necessary and sufficient condition for L^1-convergence of a modified cosine sum and a theorem of Telyakovskii [1] concerning convergence behavior of cosine series with monotonic decreasing coefficients has been deduced as a corollary.

Keywords: L^1- convergence; modified cosine sum; class S.

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1 Introduction

Consider the cosine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \ . \tag{1.1}$$

Let $S_n(x)$ denote the partial sum of (1.1) and let $f(x) = \lim_{n\to\infty} S_n(x)$.

The problem of L^1-convergence, via Fourier coefficients, consists of finding the properties of Fourier coefficients such that the necessary and sufficient condition for $\| S_n(x) - f(x) \| = o(1)$, $n \to \infty$, is given in the form $a_n \log n = o(1)$, $n \to \infty$, where $\| . \|$ denotes the L^1-norm.

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The following definitions are related to this paper.

Convex sequence. [2,vol.II,p.202] A sequence $\{a_k\}$ is said to be convex if $\Delta^2 a_k \geq 0$ for every k where $\Delta^2 a_k = \Delta a_k - \Delta a_{k+1}$ and $\Delta a_k = a_k - a_{k+1}$.

Quasi- Convex sequence [2,vol.II,p.202]. A sequence $\{a_k\}$ is said to be quasi-convex if $\sum_{k=1}^{\infty} k\,|\Delta^2 a_k| < \infty$.

The class of all such sequences is an extension of the class of convex null sequences. The class of quasi-convex sequences is a subclass of *BV* class ($\sum_{k=1}^{\infty} |\Delta a_k| < \infty$), the class of all null sequences of bounded variation.

Teljakovskii [3] generalized the notion of quasi- convexity.

Let $\{a_k\}$ be a sequence satisfying

$$a_k \to 0 \text{ as } k \to \infty ; \tag{1.2}$$

$$S_1 = \sum_{k=0}^{\infty} |\Delta a_k| < \infty ; \tag{1.3}$$

$$S_2 = \sum_{m=2}^{\infty} \left| \sum_{k=1}^{[M/2]} \frac{\Delta a_{M-k} - \Delta a_{M+k}}{k} \right| < \infty . \tag{1.4}$$

It has been established [3] that a quasi-convex null sequence satisfies the conditions (1.2) − (1.4) and imply $\lim_{n\to\infty} S_n(x)$ exists where $S_n(x)$ is the partial sum of (1.1).

Concerning L^1-convergence of the cosine series (1.1), the following theorem is known:

Theorem A [2]. If $a_k \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1.1) in the metric space L^1, it is necessary and sufficient that $a_n \log n = o(1)$, $n\to\infty$.

Teljakovskii [4] generalized Theorem A for the cosine series (1.1) with coefficients $\{a_k\}$ satisfying conditions (1.2) – (1.4) and established the following Theorem:

Theorem B [4]. Let the coefficients $\{a_k\}$ of the series (1.1) satisfy the conditions (1.2) − (1.4). If $\lim_{n\to\infty} a_n \log n = 0$, then the cosine series (1.1) converges in the L^1-metric space.

Teljakovskii [3] has also shown that under the conditions (1.2) – (1.4), the series (1.1) is a Fourier series and

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \right| dx \leq C(S_1+S_2)$$

where C is a positive constant.

Sidon generalized the concept of quasi-convexity as follows:

The class S [5]. A null sequence $\{a_k\}$ belongs to the class S if there exists a sequence $\{A_k\}$ such that (i) $A_k \downarrow 0$, $k \to \infty$, (ii) $\sum_{k=0}^{\infty} A_k < \infty$, and (iii) $|\Delta a_k| \leq A_k$ for all k .

The class S is the extension of the class of quasi-convex sequences. Since a quasi-convex null sequence satisfies conditions of the class S, if we choose $A_n = \sum_{m=n}^{\infty} |\Delta^2 a_m|$.

Teljakovskii generalized Theorem A by establishing the following theorem:

Theorem C [1]. Let $\{a_k\}$ belong to the class S. Then the cosine series (1.1) is the Fourier series of its sum f and $\| S_n(x) - f(x) \| = o(1)$, $n \to \infty$ if and only if $a_n \log n = o(1)$, $n \to \infty$.

Teljakovskii, thus showed that the class S is also a class of L^1-convergence which in turn led to numerous, more general results.

Rees and Stanojevic [6,7] introduced a new type of cosine sum

$$h_n(x = \frac{1}{2} \sum_{k=0}^{\infty} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx \tag{1.5}$$

and obtained a necessary and sufficient for its integrability.

Regarding the L^1 - convergence of (1.5) to a cosine trigonometric series belonging to the class S, Ram proved the following result:

Theorem D. [8]. If $\{a_k\}$ belongs to the class S, then

$$\| f(x) - h_n(x) \| = o(1), \quad n \to \infty .$$

Kumari and Ram [9] introduced a new modified cosine sum

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx \tag{1.6}$$

and proved the result:

Theorem E. Let $\{a_k\}$ belong to the class S.

$$\text{If } \lim_{n\to\infty} |a_{n+1}| \log n = 0, \text{ then } \| f(x) - f_n(x) \| = o(1), \ n \to \infty .$$

Hooda et al. [10] introduced new modified cosine sum

$$g_n(x) = \left(\frac{1}{2}\right)\left[a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right] + \sum_{k=1}^{n} \left[a_{n+1} + \sum_{j=k}^{n} \Delta^2 a_j\right] \cos kx, \tag{1.7}$$

and studied the necessary and sufficient conditions for the L^1-convergence and integrability of the limit of (1.7) under the conditions (1.2) – (1.4).

In recent years, significant results have been developed by various authors [11-16] by imposing different conditions on the coefficients a_k of trigonometric series (1.1). The aim of this paper is to study the L^1-convergence of (1.7) under class S on the coefficients a_k and deduce Theorem C as a corollary of our result.

2 Lemma

The following lemma is required for the proof of our result:

Lemma 1. [17]. If $|c_k| \leq 1$, then

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k D_k(x) \right| dx \leq C\,(n+1),$$

where C is a positive constant and $D_n(x) = (1/2) + \cos x + \ldots + \cos nx$ represents Dirichlet's kernel.

3 Results

Theorem 1. Let $\{a_k\}$ belong to the class S, then

$\| f(x) - g_n(x) \| = o(1),\ n \to \infty$ if and only if $a_n \log n = o(1),\ n \to \infty$.

Corollary 1. Let $\{a_k\}$ belong to the class S, then

$\| S_n(x) - f(x) \| = o(1),\ n \to \infty$ if and only if $a_n \log n = o(1),\ n \to \infty$. This is nothing but theorem C.

Proof of Theorem 1. We have

$$g_n(x) = \left(\frac{1}{2}\right)\left[a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right] + \sum_{k=1}^{n} \left[a_{n+1} + \sum_{j=k}^{n} \Delta^2 a_j\right] \cos kx$$

$$= (1/2)[a_0 - a_{n+1} + a_{n+2}] + \sum_{k=1}^{n} [a_k - a_{n+1} + a_{n+2}] \cos kx$$

$$= (a_0/2) - (1/2)\Delta a_{n+1} + \sum_{k=1}^{n} a_k \cos kx - \Delta a_{n+1} \sum_{k=1}^{n} \cos kx$$

$$= (a_0/2) + \sum_{k=1}^{n} a_k \cos kx - \Delta a_{n+1} \sum_{K=1}^{n} \left[\cos ks + \frac{1}{2}\right]$$

$$= S_n(x) - \Delta a_{n+1} D_n(x).$$

Using Abel's transformation, we get

$$g_n(x) = \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - \Delta a_{n+1} D_n(x) \tag{3.1}$$

$$= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+2} D_n(x).$$

Now,

$$f(x) - g_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+2} D_n(x).$$

Abel transformation with lemma1 yield,

$$\int_0^{\pi} \left| f(x) - g_n(x) \right| dx$$

$$\leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} \left| a_{n+2} D_n(x) \right| dx$$

$$= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x) \right| dx + \int_0^{\pi} \left| a_{n+2} D_n(x) \right| dx$$

$$= \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta A_k \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j(x) \right| dx + + \int_0^{\pi} \left| a_{n+2} D_n(x) \right| dx$$

$$\leq C \sum_{k=n+1}^{\infty} (k+1) \Delta A_k + \int_0^{\pi} \left| a_{n+2} D_n(x) \right| dx .$$

Now, $\int_0^{\pi} \left| a_{n+2} D_n(x) \right|$ behaves like $a_n \log n$ and under the assumed hypothesis $a_n \log n = o(1)$, $n \longrightarrow \infty$ as well as $\sum_{k=n+1}^{\infty} (k+1) \Delta A_k$ converges, the right hand side tends to zero as $n \longrightarrow \infty$ and this gives

$$\lim_{n\to\infty} \int_0^{\pi} \left| f(x) - g_n(x) \right| dx = 0.$$

On the other hand,

$$a_{n+2} D_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - [f(x) + g_n(x)] ,$$

and so

$$\int_0^{\pi} \left| a_{n+2} D_n(x) \right| dx \leq \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx + \int_0^{\pi} \left| f(x) - g_n(x) \right| dx .$$

Using the hypothesis of the theorem along with above estimates, the right hand side tends to zero as $n \longrightarrow \infty$.

This completes the proof of our theorem.

Proof of Corollary 1. We have

$$\int_0^{\pi} | f(x) - S_n(x) | \, dx = \int_0^{\pi} | f(x) - g_n(x) + g_n(x) - S_n(x) | \, dx$$
$$\leq \int_0^{\pi} | f(x) - g_n(x) | dx + \int_0^{\pi} | g_n(x) - S_n(x) | dx$$
$$\leq \int_0^{\pi} | f(x) - f_n(x) | dx + \int_0^{\pi} | \Delta a_{n+1} D_n(x) | dx$$

whereas

$$\int_0^{\pi} | \Delta a_{n+1} D_n(x) | dx \leq \int_0^{\pi} | f(x) - f_n(x) | dx + \int_0^{\pi} | f(x) - S_n(x) | \, dx .$$

Since $\int_0^{\pi} | D_n(x) | dx$ behave like log n for large values of n and by the hypothesis of our result the corollary follows.

4 Conclusion

In this paper, a new approach has been developed to obtain a necessary and sufficient condition for L^1–convergence of trigonometric series (1.1). Our results can be generalized to obtain more interesting results.

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Competing Interests

Author has declared that no competing interests exist.

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